4a. In ❶-❷, if there exists a sequence $\{a_n\}_{n=1}^{\infty}$ satisfying the given conditions, write its n^{th} term in the box; and if no such sequence exists, write DOES NOT EXIST in the box. No explanation is required.

❶ $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ and $\lim_{n\to\infty} a_n$ does not exist.

$a_n = \boxed{n}$

❷ $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = -1$ and $\lim_{n\to\infty} a_n$ exists.

$a_n = \boxed{\frac{(-1)^n}{n}}$

4b. Let c be a real number, and consider the sequence $\{a_n\}_{n=1}^{\infty}$ with $a_1 = c$ and satisfying the recursion relation $a_{n+1} = a_n + a_n^2$ for all $n \geq 1$.

① Show that if the sequence converges, then $\lim_{n\to\infty} a_n = 0$.

Suppose $L = \lim_{n\to\infty} a_n$. Then:

$a_{n+1} = a_n + a_n^2$ for all $n \geq 1 \Rightarrow L = L + L^2 \Rightarrow L^2 = 0 \Rightarrow L = 0$

② Fill in the boxes so that the sentence below becomes a true statement.

If $c = \boxed{\frac{1}{2}}$, *then the sequence* $\boxed{\text{diverges}}$.

Write **here** a real number which is <u>not</u> an integer

Write **here** either *converges* or *diverges*

③ Prove the statement in ②.

$a_1 = \frac{1}{2} \geq \frac{1}{2}$ and if $a_n \geq \frac{1}{2}$, then $a_{n+1} = a_n + a_n^2 \geq \frac{1}{2} + 0 = \frac{1}{2}$.

Hence, by induction, $a_n \geq \frac{1}{2}$ for all $n \geq 1$.

It follows that $\lim_{n\to\infty} a_n \geq \frac{1}{2}$ if the limit exists,

Contradicting Part①. Therefore the limit does not exist.