

2. The Tricomi equation

$$yu_{xx} + u_{yy} = 0$$

arises in the study of transonic flow in fluid mechanics and in the study of isometric embeddings of 2-dimensional Riemannian manifolds into 3-dimensional Euclidian space in differential geometry.

Find all possible values of the pair of constants (a, b) for which the function $u(x, y) = (ax^2 + y^3)^b$ satisfies the Tricomi equation for all (x, y) with $ax^2 + y^3 > 0$.

$$u_x = b(ax^2 + y^3)^{b-1} \cdot 2ax$$

$$u_{xx} = b \cdot (b-1)(ax^2 + y^3)^{b-2} \cdot (2ax)^2 + b(ax^2 + y^3)^{b-1} \cdot 2a$$

$$u_y = b(ax^2 + y^3)^{b-1} \cdot 3y^2$$

$$u_{yy} = b \cdot (b-1)(ax^2 + y^3)^{b-2} \cdot (3y^2)^2 + b(ax^2 + y^3)^{b-1} \cdot 6y$$

$$yu_{xx} + u_{yy} = by(ax^2 + y^3)^{b-2} \cdot \left((b-1)(4a^2x^2 + 9y^3) + (ax^2 + y^3) \cdot (2a+6) \right)$$

$$= by(ax^2 + y^3)^{b-2} \cdot \left(\{ (b-1) \cdot 4a^2 + 2(a+3) \cdot a \} x^2 + \{ 9(b-1) + 2(a+3) \} y^3 \right)$$

$$yu_{xx} + u_{yy} = 0 \text{ for all } (x, y) \text{ with } ax^2 + y^3 > 0$$

$$\Leftrightarrow b=0 \text{ or } \left(4a^2(b-1) + 2a(a+3) = 0 \text{ and } 9(b-1) + 2(a+3) = 0 \right)$$

$$a(4a-9)(b-1) = 0 \Rightarrow \begin{array}{l} a=0 \\ \Downarrow \\ b = \frac{1}{3} \end{array} \text{ or } \begin{array}{l} a = \frac{9}{4} \\ \Downarrow \\ b = -\frac{1}{6} \end{array} \text{ or } \begin{array}{l} b=1 \\ \Downarrow \\ a = -3 \end{array}$$

u satisfies the Tricomi equation exactly when

$$(a, b) = \left(0, \frac{1}{3} \right), \left(\frac{9}{4}, -\frac{1}{6} \right), (-3, 1), (a, 0)$$

\uparrow
a arbitrary