5. Consider the power series $f(x) = \sum_{n=2}^{\infty} \frac{x^n}{n^3 - n}$.

a. Find the interval of convergence I of the power series and determine whether it converges absolutely or conditionally at each point of I.

$$c_n = \frac{1}{n^3-n} \Rightarrow \frac{1}{R} = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^3-(n+1)}}{\frac{1}{n^3-n}} = \lim_{n\to\infty} \frac{n-1}{n+2} = 1 \Rightarrow R = 1$$

$\underline{x=1} \Rightarrow f(1) = \sum_{n=2}^{\infty} \frac{1}{n^3-n}$ converges by LCT because

$c = \lim_{n\to\infty} \frac{\frac{1}{n^3-n}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = 1 < \infty$ and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

(p-series with $p = 3 > 1$)

$\underline{x=-1} \Rightarrow f(-1) = \sum_{n=2}^{\infty} \frac{(-1)^n}{n^3-n}$ converges absolutely because

$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n^3-n} \right| = \sum_{n=2}^{\infty} \frac{1}{n^3-n}$ converges.

So: $f(x)$ converges absolutely at every point of its interval of convergence $I = [-1, 1]$.

b. Find the exact value of $f(-1)$.

$$f(-1) = \sum_{n=2}^{\infty} \frac{(-1)^n}{n^3-n} = \sum_{n=2}^{\infty} (-1)^n \left(\frac{n}{n^2-1} - \frac{1}{n} \right) = \sum_{n=2} (-1)^n \left(\frac{1}{2} \cdot \frac{1}{n-1} + \frac{1}{2} \cdot \frac{1}{n+1} - \frac{1}{n} \right)$$

$$= \frac{1}{2} \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1} + \frac{1}{2} \sum_{n=2}^{\infty} \frac{(-1)^n}{n+1} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n}$$

$$= \frac{1}{2} \cdot \ln 2 + \frac{1}{2} \cdot \left(\ln 2 - \left(1 - \frac{1}{2}\right) \right) + \ln 2 - 1 = 2 \ln 2 - \frac{5}{4}$$