1. Consider the function

$$f(x,y,z) = x^3y^2z + ax^2y + bxz^2\,,$$

where a and b are constants, the point $P_0(1,-1,2)$, and the vector $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$.

a. Compute $\nabla f(P_0)$.

$$\vec{\nabla} f = (3x^2y^2z + 2axy + bz^2)\vec{i} + (2x^3yz + ax^2)\vec{j} + (x^3y^2 + 2bxz)\vec{k}$$

$$\vec{\nabla} f(P_0) = (6-2a+4b)\vec{i} + (-4+a)\vec{j} + (1+4b)\vec{k}$$

b. Find all possible values of (a,b) for which f increases the fastest in the direction of $\mathbf{A}$ at P_0.

We want $\vec{\nabla} f(P_0) = c \cdot \vec{A}$ with $c > 0$.

$$\vec{\nabla} f(P_0) \,/\!/\, \vec{A} \Rightarrow \frac{6-2a+4b}{2} = \frac{-4+a}{3} = \frac{1+4b}{6}$$

$$\Rightarrow \left\{\begin{matrix} -8a+12b=-26 \\ 6a-12b=27 \end{matrix}\right\} \Rightarrow a = -\frac{1}{2} \text{ and } b = -\frac{5}{2}$$

$$\Rightarrow \vec{\nabla} f(P_0) = -3\vec{i} - \frac{9}{2}\vec{j} - 9\vec{k} = -\frac{3}{2}\cdot\vec{A} \text{ and } c = -\frac{3}{2} < 0$$

$\Rightarrow$ There is no such (a,b)

c. Let $a = 3$ and $b = 1$. Find the directional derivative of f in the direction of $\mathbf{A}$ at P_0.

$$\vec{\nabla} f = 4\vec{i} - \vec{j} + 5\vec{k}$$

$$\vec{u} = (\text{the direction of } \vec{A}) = \frac{2\vec{i}+3\vec{j}+6\vec{k}}{\sqrt{2^2+3^2+6^2}} = \frac{2\vec{i}+3\vec{j}+6\vec{k}}{7}$$

(The directional derivative of f at P_0 in the direction of $\vec{A}$)

$$= D_{\vec{u}} f(P_0) = \vec{\nabla} f(P_0)\cdot\vec{u} = (4\vec{i}-\vec{j}+5\vec{k})\cdot\frac{2\vec{i}+3\vec{j}+6\vec{k}}{7} = 5$$