4. Find the absolute maximum and minimum values of the restriction of the function

$$f(x, y, z) = xy + xz$$

to the unit sphere $\underbrace{x^2 + y^2 + z^2}_{g(x,y,z)} = 1$.

$$\left.\begin{array}{l} \vec{\nabla} f = \lambda \vec{\nabla} g \\ g = c \end{array}\right\} \Rightarrow \left\{\begin{array}{l} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g = c \end{array}\right\} \Rightarrow \left\{\begin{array}{ll} y + z = \lambda \cdot 2x & (1) \\ x = \lambda \cdot 2y & (2) \\ x = \lambda \cdot 2z & (3) \\ x^2 + y^2 + z^2 = 1 & (4) \end{array}\right.$$

(2) and (3) $\Rightarrow$ $\lambda y = \lambda z$ $\Downarrow$ (5) $y = z$ or $\lambda = 0$

(5) $y = z \overset{(1)}{\Downarrow}$ (6) $y = \lambda x \overset{(2)}{\Downarrow}$ $x = 2\lambda^2 x$ $\Downarrow$ $x = 0$ or $\lambda = \frac{1}{\sqrt{2}}$ or $\lambda = -\frac{1}{\sqrt{2}}$

$x = 0 \overset{(6)}{\Downarrow} y = 0 \overset{(5)}{\Downarrow} z = 0 \overset{(4)}{\Rightarrow} 0 = 1$

no solution

$\lambda = \frac{1}{\sqrt{2}} \overset{(2)}{\Downarrow} x = \sqrt{2}\, y$

$\lambda = -\frac{1}{\sqrt{2}} \overset{(2)}{\Downarrow} x = -\sqrt{2}\, y$

(4), (5) $\Rightarrow$ $4y^2 = 1$ $\Downarrow$ $y = \pm\frac{1}{2} \overset{(5)}{\Downarrow}$ $z = \pm\frac{1}{2}$

$x = \pm\frac{1}{\sqrt{2}}$, $\quad x = \mp\frac{1}{\sqrt{2}}$

$\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2}, \pm\frac{1}{2}\right) \to \boxed{1/\sqrt{2}}$

$\left(\mp\frac{1}{\sqrt{2}}, \pm\frac{1}{2}, \pm\frac{1}{2}\right) \to \boxed{-1/\sqrt{2}}$

$\lambda = 0 \overset{(1)}{\Downarrow} z = -y$; $\quad \lambda = 0 \overset{(2)}{\Rightarrow} x = 0$

$x = 0,\ z = -y \overset{(4)}{\Rightarrow} 2y^2 = 1 \Downarrow y = \pm\frac{1}{\sqrt{2}}$, $\quad z = \mp\frac{1}{\sqrt{2}}$

$\left(0, \pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\right) \to \boxed{0}$

As the unit sphere is bounded,

abs max is $\frac{1}{\sqrt{2}}$ and abs min is $-\frac{1}{\sqrt{2}}$.