4. Consider the power series $f(x) = \sum_{n=2}^{\infty} \frac{x^n}{(n^2-1)\cdot n!}$.

a. Find the radius of convergence R of the power series.

$$c_n = \frac{1}{(n^2-1)\cdot n!} = \frac{1}{(n-1)\cdot(n+1)!} \quad \text{for } n \geq 2$$

$$\frac{1}{R} = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = \lim_{n\to\infty} \frac{\frac{1}{n\cdot(n+2)!}}{\frac{1}{(n-1)\cdot(n+1)!}} = \lim_{n\to\infty} \frac{(n-1)\cdot(n+1)!}{n\cdot(n+2)!}$$

$$= \lim_{n\to\infty} \frac{n-1}{n\cdot(n+2)} = 0 \Rightarrow R = \infty$$

b. Show that $f(1) < \frac{1}{2}e - \frac{7}{6}$.

$$f(1) = \sum_{n=2}^{\infty} \frac{1}{(n-1)\cdot(n+1)!} = \frac{1}{3!} + \sum_{n=3}^{\infty} \frac{1}{(n-1)\cdot(n+1)!} < \frac{1}{3!} + \frac{1}{2}\sum_{n=3}^{\infty} \frac{1}{(n+1)!}$$

$$= \frac{1}{3!} + \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{1}{n!} - 1 - 1 - \frac{1}{2!} - \frac{1}{3!}\right) = \frac{1}{2}e - \frac{7}{6}$$

c. Show that $f(-1) < \frac{3}{20}$.

$$f(-1) = \sum_{n=2}^{\infty} \frac{(-1)^n}{(n-1)\cdot(n+1)!} < \frac{1}{3!} - \frac{1}{2\cdot 4!} + \frac{1}{3\cdot 5!} = \frac{107}{720} < \frac{3}{20}$$

by the Alternating Series Estimate which is applicable as $\left\{\frac{1}{(n-1)\cdot(n+1)!}\right\}_{n=2}^{\infty}$ is a decreasing sequence with limit 0.