1. Consider the sequence $\{a_n\}_{n=1}^{\infty}$ satisfying the conditions $a_1 = A$ and $a_{n+1} = 3a_n - \frac{1}{a_n}$ for $n \geq 1$ where A is a real number such that $a_n \neq 0$ for all $n \geq 1$.

a. Assume that the sequence converges and let $L = \lim_{n\to\infty} a_n$. Show that, depending on A, there are at most two possible values for L.

$$a_{n+1} = 3a_n - \frac{1}{a_n} \text{ for all } n \geq 1 \xRightarrow[n\to\infty]{} L = 3L - \frac{1}{L} \Rightarrow L^2 = \frac{1}{2}$$

$$\Rightarrow L = \frac{1}{\sqrt{2}} \text{ or } L = -\frac{1}{\sqrt{2}}$$

b. Give an example of A for which the sequence converges. Explain your reasoning.

Since $3 \cdot \frac{1}{\sqrt{2}} - \frac{1}{1/\sqrt{2}} = \frac{1}{\sqrt{2}}$, if $A = \frac{1}{\sqrt{2}}$, then $a_n = \frac{1}{\sqrt{2}}$ for all $n \geq 1$, and the sequence converges.

c. Give an example of A for which the sequence diverges and $|a_n| < 1$ for all $n \geq 1$. Explain your reasoning.

Since $3 \cdot \frac{1}{2} - \frac{1}{1/2} = -\frac{1}{2}$ and $3 \cdot \left(-\frac{1}{2}\right) - \frac{1}{-1/2} = \frac{1}{2}$, if $A = \frac{1}{2}$, then $a_n = \frac{1}{2}$ for odd n and $-\frac{1}{2}$ for even n, and the sequence diverges.

d. Show that the sequence is increasing if $A = 1$.

- $a_2 = 3a_1 - \frac{1}{a_1} = 3 \cdot 1 - \frac{1}{1} = 2 > 1 = a_1 > 0$
- If $0 < a_k < a_{k+1}$ for some $k \geq 1$, then $\frac{1}{a_k} > \frac{1}{a_{k+1}} \Rightarrow -\frac{1}{a_k} < -\frac{1}{a_{k+1}}$ and $3a_k < 3a_{k+1}$, and hence $0 < a_{k+1} = 3a_k - \frac{1}{a_k} < 3a_{k+1} - \frac{1}{a_{k+1}} = a_{k+2}$

Therefore $0 < a_n < a_{n+1}$ for all $n \geq 1$ by induction.

e. Determine whether the sequence converges or diverges if $A = 1$.

By Part(d), $a_n \geq 1$ for all $n \geq 1$. If the sequence converges, then $L = \lim_{n\to\infty} a_n \geq 1$, contradicting Part(a). Hence the sequence diverges.