2. The *Dym equation*

$$u_t = u^3 u_{xxx}$$

is a nonlinear evolution equation which arises in the study of the motion of the interface between a viscous and a nonviscous fluid with surface tension.

Find all nonzero constants a, b, c such that the function

$$u(x,t) = (ax+bt)^c$$

satisfies the Dym equation for all (x,t) with $ax+bt > 0$.

$$u_t = c\cdot(ax+bt)^{c-1}\cdot b$$

$$u_x = c\cdot(ax+bt)^{c-1}\cdot a$$

$$u_{xx} = c\cdot(c-1)\cdot(ax+bt)^{c-2}\cdot a^2$$

$$u_{xxx} = c\cdot(c-1)\cdot(c-2)\cdot(ax+bt)^{c-3}\cdot a^3$$

$$u^3 u_{xxx} = c\cdot(c-1)\cdot(c-2)\cdot(ax+bt)^{4c-3}\cdot a^3$$

$$u_t = u^3\cdot u_{xxx}$$

$$\Downarrow$$

$$bc\,(ax+bt)^{c-1} = a^3 c(c-1)(c-2)(ax+bt)^{4c-3}$$

$$\Downarrow \text{ as } c\neq 0$$

$$b = a^3(c-1)(c-2)(ax+bt)^{3c-2}$$

As $b\neq 0$, this is possible only if $3c-2=0$.

So $\boxed{c=\frac{2}{3}}$ and $\boxed{b=\frac{4}{9}a^3}$.

The most general solution of this form is

$$u(x,t) = \left(ax+\frac{4}{9}a^3 t\right)^{2/3}$$

where a is an arbitrary nonzero constant.