

4. Let  $R(a)$  be the region bounded by the graph of  $f(x) = ax - x^2$  and the  $x$ -axis for  $0 \leq x \leq a$ , where  $a$  is a positive constant.

a. Compute the volume  $V(a)$  of the solid generated by revolving  $R(a)$  about the  $x$ -axis.

$$\begin{aligned} V(a) &= \pi \int_0^a (ax - x^2)^2 dx = \pi \int_0^a (a^2x^2 - 2ax^3 + x^4) dx \\ &= \pi \left[ \frac{a^2}{3} x^3 - \frac{a}{2} x^4 + \frac{1}{5} x^5 \right]_0^a = \pi \left( \frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) a^5 = \frac{\pi}{30} a^5 \end{aligned}$$

b. Compute the volume  $W(a)$  of the solid generated by revolving  $R(a)$  about the  $y$ -axis.

$$\begin{aligned} W(a) &= 2\pi \int_0^a x \cdot (ax - x^2) dx = 2\pi \int_0^a (ax^2 - x^3) dx \\ &= 2\pi \left[ \frac{a}{3} x^3 - \frac{1}{4} x^4 \right]_0^a = 2\pi \left( \frac{1}{3} - \frac{1}{4} \right) a^4 = \frac{\pi}{6} a^4 \end{aligned}$$

c. Find all values of  $a$  for which  $V(a) = W(a)$ .

$$V(a) = W(a) \Rightarrow \frac{\pi}{30} a^5 = \frac{\pi}{6} a^4 \Rightarrow a = 5$$