2a. Suppose that f is a continuous function satisfying

$$f(x) = x - x^2 - x\int_0^x f(t)\,dt \qquad (*)$$

for all x, and c is a real number such that $f'(c) = 0$. Express $f(c)$ in terms of c only.

$$(*) \xRightarrow{d/dx} f'(x) = 1 - 2x - \int_0^x f(t)\,dt - x f(x) \quad \text{by FTC 1}$$

$$\Downarrow x=c$$

$$0 = f'(c) = 1 - 2c - \int_0^c f(t)\,dt - c f(c) \qquad (1)$$

$$(*) \xRightarrow{x=c} f(c) = c - c^2 - c\int_0^c f(t)\,dt. \qquad (2)$$

$-c \times (1) + (2)$: $\quad f(c) = c^2 + c^2 f(c)$

$$\Downarrow$$

$$f(c) = \frac{c^2}{1-c^2}$$

2b. Evaluate the limit $\displaystyle\lim_{n\to\infty} \frac{1}{n^3}\left(\sum_{i=1}^{3n}\sqrt{i}\right)^2$.

$\displaystyle\frac{1}{n^{3/2}}\sum_{i=1}^{3n}\sqrt{i} = \sum_{i=1}^{3n}\sqrt{\frac{i}{n}}\cdot\frac{1}{n}$ is a right Rieman sum for $f(x)=\sqrt{x}$ over the interval $[0,3]$.

$$\Rightarrow \lim_{n\to\infty}\frac{1}{n^{3/2}}\sum_{i=1}^{3n}\sqrt{i} = \int_0^3 \sqrt{x}\,dx = \left.\frac{x^{3/2}}{3/2}\right]_0^3 = \frac{2}{3}\cdot 3^{3/2} = 2\cdot 3^{1/2}$$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{n^3}\left(\sum_{i=1}^{3n}\sqrt{i}\right)^2 = \left(\lim_{n\to\infty}\frac{1}{n^{3/2}}\sum_{i=1}^{3n}\sqrt{i}\right)^2 = (2\cdot 3^{1/2})^2 = 12$$