3. A function f with a continuous second derivative satisfies:

$$\int_0^{\pi/2} \sin(x) f(x)\,dx \overset{①}{=} 1 \qquad \int_0^{\pi/2} \sin(x) f'(x)\,dx \overset{③}{=} 3 \qquad \int_0^{\pi/2} \sin(2x) f(x)\,dx \overset{⑤}{=} 5$$

$$\int_0^{\pi/2} \cos(x) f(x)\,dx \overset{②}{=} 2 \qquad \int_0^{\pi/2} \cos(x) f'(x)\,dx \overset{④}{=} 4 \qquad \int_0^{\pi/2} \cos(2x) f(x)\,dx \overset{⑥}{=} 6$$

Evaluate $\displaystyle\int_0^{\pi/2} \sin(2x) f''(x)\,dx$.

$$\int_0^{\pi/2} \sin(2x) f''(x)\,dx = \int_0^{\pi/2} \sin(2x)\,d(f'(x)) = \Big[\sin(2x) f'(x)\Big]_0^{\pi/2} - \int_0^{\pi/2} f'(x)\,d(\sin(2x))$$

$$= \underbrace{\sin(\pi)}_{0} f'(\tfrac{\pi}{2}) - \underbrace{\sin(0)}_{0} f'(0) - 2\int_0^{\pi/2} \cos(2x) f'(x)\,dx = -2\int_0^{\pi/2} \cos(2x)\,d(f(x))$$

$$= -2\Big[\cos(2x) f(x)\Big]_0^{\pi/2} + 2\int_0^{\pi/2} f(x)\,d(\cos(2x))$$

$$= -2\underbrace{\cos(\pi)}_{-1}\underbrace{f(\tfrac{\pi}{2})}_{5} + 2\underbrace{\cos(0)}_{1}\underbrace{f(0)}_{-3} - 4\underbrace{\int_0^{\pi/2} \sin(2x) f(x)\,dx}_{5 \text{ by } ⑤} = 10-6-20=-16$$

because ③ + ② gives:

$$5 = 3+2 = \int_0^{\pi/2} (\sin(x) f'(x) + \cos(x) f(x))\,dx = \int_0^{\pi/2} d(\sin(x) f(x))$$

$$= \Big[\sin(x) f(x)\Big]_0^{\pi/2} = \underbrace{\sin(\tfrac{\pi}{2})}_{1} f(\tfrac{\pi}{2}) - \underbrace{\sin(0)}_{0} f(0) = f(\tfrac{\pi}{2})$$

and ④ − ① gives:

$$3 = 4-1 = \int_0^{\pi/2} (\cos(x) f'(x) - \sin(x) f(x))\,dx = \int_0^{\pi/2} d(\cos(x) f(x))$$

$$= \Big[\cos(x) f(x)\Big]_0^{\pi/2} = \underbrace{\cos(\tfrac{\pi}{2})}_{0} f(\tfrac{\pi}{2}) - \underbrace{\cos(0)}_{1} f(0) = -f(0)$$