4. In each of the following, if the given statement is true, then mark the ☐ to the left of TRUE with a ✗ and prove the statement; otherwise, mark the ☐ to the left of FALSE with a ✗ and give a counterexample.

a. If f is differentiable on $(0,\infty)$ and $f(1/x) = f(x)$ for all $x > 0$, then there is a c in $(0,\infty)$ such that $f'(c) = 0$.

☒ TRUE ☐ FALSE

$$f(1/x) = f(x) \Rightarrow f'(1/x)\cdot(-1/x^2) = f'(x) \xRightarrow{\boxed{x=1}} -f'(1) = f'(1) \Rightarrow f'(1) = 0$$

b. If f is differentiable on $(0,\infty)$ and $f(x^2) = (f(x))^3$ for all $x > 0$, then there is a c in $(0,\infty)$ such that $f'(c) = 0$.

☒ TRUE ☐ FALSE

$$f(x^2) = f(x)^3 \xRightarrow{\boxed{x=1}} f(1) = f(1)^3 \Rightarrow f(1) \text{ is } 1, 0 \text{ or } -1$$

$$\Downarrow$$

$$f'(x^2)\cdot 2x = 3f(x)^2\cdot f'(x) \xRightarrow{\boxed{x=1}} 2f'(1) = 3f(1)^2 f'(1) \Rightarrow (3f(1)^2 - 2)\cdot f'(1) = 0 \Rightarrow f'(1) = 0$$

c. If f is differentiable on $(0,\infty)$ and $f(x)f(2x) \geq 0$ for all $x > 0$, then there is a c in $(0,\infty)$ such that $f'(c) = 0$.

☐ TRUE ☒ FALSE

Let $f(x) = x$.

Then $f(x)f(2x) = x\cdot 2x = 2x^2 \geq 0$ for all $x > 0$,

but $f'(x) = 1 \neq 0$ for all x.

d. If f is differentiable on $(0,\infty)$ and $f(x)f(2x) \leq 0$ for all $x > 0$, then there is a c in $(0,\infty)$ such that $f'(c) = 0$.

☒ TRUE ☐ FALSE

f is diff'ble on $(0,\infty) \Rightarrow f$ is continuous on $(0,\infty)$.

- If $f(1)f(2) < 0$, then applying IVT to f on $[1,2]$ we conclude that f has a zero in $(1,2)$
- Otherwise, $f(1)f(2) = 0$ and f has a zero at 1 or 2.

Therefore, in both cases there is a in $[1,2]$ such that $f(a) = 0$.

Similarly, there is b in $[3,6]$ such that $f(b) = 0$.

Applying Rolle's Theorem to f on $[a,b]$ we conclude that there is c in (a,b) such that $f'(c) = 0$.