3. Consider the function

$$f(x) = \exp(-(\ln x)^k) = e^{-((\ln x)^k)}$$

where k is a constant.

a. In this part, let $k = 3$ and compute $f'(e^2)$.

$$f(x) = e^{-(\ln x)^3} \Rightarrow f'(x) = e^{-(\ln x)^3} \cdot \left(-3(\ln x)^2 \cdot \frac{1}{x}\right)$$

$$\Rightarrow f'(e^2) = e^{-(\ln(e^2))^3} \cdot \left(-3 \cdot (\ln(e^2))^2 \cdot \frac{1}{e^2}\right) = e^{-2^3} \cdot \left(-3 \cdot 2^2 \cdot \frac{1}{e^2}\right) = -\frac{12}{e^{10}}$$

b. In this part, let $k = 2$ and compute $\displaystyle\int_1^\infty f(x)\frac{\ln x}{x}\,dx$. [Here you may use the fact that $\displaystyle\int_a^\infty e^{-x}\,dx = e^{-a}$.]

$$\int_1^\infty f(x) \cdot \frac{\ln x}{x}\,dx = \int_1^\infty e^{-(\ln x)^2} \cdot \frac{\ln x}{x}\,dx = \int_0^\infty e^{-u} \cdot \frac{1}{2}\,du = \frac{1}{2}e^0 = \frac{1}{2}$$

$$\boxed{\begin{array}{l} u = (\ln x)^2 \\ du = 2\ln x \cdot \frac{1}{x}\,dx \end{array}}$$

c. In this part, assume that $f(x^2) = (f(x))^3$ for all $x > 1$ and find k.

$$f(x^2) = f(x)^3 \text{ for all } x > 1 \iff e^{-(\ln(x^2))^k} = \left(e^{-(\ln x)^k}\right)^3 \text{ for all } x > 1$$

$$\iff e^{-(2\ln x)^k} = e^{-3(\ln x)^k} \text{ for all } x > 1 \iff 2^k (\ln x)^k = 3(\ln x)^k \text{ for all } x > 1$$

$$\iff 2^k = 3 \iff k = \log_2 3$$