Final Exam Question 4.

> *Do not forget to write your **full name** and your **Bilkent ID number**, and **sign** on the upper right corner of your paper.*

For $t \geq 0$, let $A(t)$ denote the area of the triangle formed by the coordinate axes and the tangent line to the curve $y = e^{-x}$ at the point (t, e^{-t}). Find the absolute maximum and minimum values of A.

Show all your work!
Explain your reasoning fully and in detail using correct mathematical notation and terminology, and in well-formed mathematical and English sentences!

$y' = -e^{-x} \Rightarrow y'|_{x=t} = -e^{-t}$

An equation of the tangent line is:

$$y - e^{-t} = -e^{-t} \cdot (x-t)$$

x-intercept: $y=0 \Rightarrow -e^{-t} = -e^{-t} \cdot (x-t) \Rightarrow x = t+1$

y-intercept: $x=0 \Rightarrow y - e^{-t} = e^{-t} \cdot t \Rightarrow y = (t+1)e^{-t}$

Hence: $A = \frac{1}{2} \cdot (t+1) \cdot (t+1)e^{-t} = \frac{1}{2}(t+1)^2 e^{-t}$ for $0 \leq t < \infty$.

Critical points: $\frac{dA}{dt} = (t+1)e^{-t} - \frac{1}{2}(t+1)^2 e^{-t} = \frac{1}{2}(1-t^2)e^{-t} = 0 \Rightarrow t = 1$ or $t = -1$ $\otimes$

$\Rightarrow t = 1 \Rightarrow A = 2e^{-1} = \frac{2}{e}$

"Endpoints":

$t = 0 \Rightarrow A = \frac{1}{2}$

$t = \infty \Rightarrow \lim_{t\to\infty} A = \lim_{t\to\infty} \frac{(t+1)^2}{2e^t} \overset{L'H}{=} \lim_{t\to\infty} \frac{2(t+1)}{2e^t} \overset{L'H}{=} \lim_{t\to\infty} \frac{1}{e^t} = 0$

As $3 > e$, $\frac{2}{e} > \frac{1}{2} > 0$, and the absolute maximum is $\frac{2}{e}$.

As 0 occurs only as a limit, there is no absolute minimum.