Do not forget to write your **full name** and your **Bilkent ID number**, and **sign** on the upper right corner of your paper.

Final Exam Question 3.

Let $R(a)$ be the region between the curve $y = x^{1/2}(9-x)^{1/3}$ and the x-axis for $0 \le x \le a$.
Let $V(a)$ be the volume of the solid generated by revolving $R(a)$ about the x-axis.
Let $W(a)$ be the volume of the solid generated by revolving $R(a)$ about the y-axis.

a. Compute $V(9)$.

b. Find all $a \ge 0$ for which $\dfrac{dV}{da} = \dfrac{dW}{da}$.

Show all your work!
Explain your reasoning fully and in detail using correct mathematical notation and terminology, and in well-formed mathematical and English sentences!

a. $V(9) = \pi \int_0^9 \left(x^{1/2}(9-x)^{1/3}\right)^2 dx = \pi \int_0^9 x\cdot(9-x)^{2/3}\,dx$

$= \pi \int_9^0 (9-u)\cdot u^{2/3}\cdot(-du) = \pi \int_0^9 (9u^{2/3} - u^{5/3})\,du = \pi\left[9\cdot\frac{u^{5/3}}{5/3} - \frac{u^{8/3}}{8/3}\right]_0^9$

$u = 9-x$, $du = -dx$

$= 3\pi\left(\frac{9\cdot 9^{5/3}}{5} - \frac{9^{8/3}}{8}\right) = 3\pi\cdot\frac{3}{40}\cdot 9^{8/3} = \frac{\pi}{40}\cdot 3^7\cdot\sqrt[3]{3} = \frac{2187\pi}{40}\sqrt[3]{3}$

b. $V = \pi \int_0^a \left(x^{1/2}(9-x)^{1/3}\right)^2 dx$ and $W = 2\pi \int_0^a x\cdot\left|x^{1/2}(9-x)^{1/3}\right| dx$

$V' = W' \overset{FTC1}{\Longrightarrow} \pi\left(a^{1/2}(9-a)^{1/3}\right)^2 = 2\pi\, a\left|a^{1/2}(9-a)^{1/3}\right| \Rightarrow a(9-a)^{2/3} = 2a^{3/2}|9-a|^{1/3}$

$\Rightarrow$ $a=0$ or $9-a=0$ or $|9-a|^{1/3} = 2a^{1/2}$

$\Downarrow$

$(9-a)^2 = 2^6 a^3 \Rightarrow \underbrace{64a^3 - a^2 + 18a - 81}_{(a-1)\cdot(64a^2+63a+81)} = 0 \Rightarrow a = 1$

Hence $a = 0, 1, 9$.