3a. The slope of the tangent line at each point (x, y) on the graph of a differentiable function $y = f(x)$ is proportional to $x^2 - 5$. If $f(1) = 1$ and $f(3) = 3$, find $f(2)$.

$f'(x) = k \cdot (x^2 - 5)$ for some constant k

$\Downarrow$

$f(x) = \int f'(x)\,dx = k \int (x^2 - 5)\,dx = k \cdot \left(\frac{x^3}{3} - 5x\right) + C$

$$\left.\begin{cases} 1 = f(1) = k \cdot \left(\frac{1}{3} - 5\right) + C = -\frac{14}{3}k + C \\ 3 = f(3) = k \cdot (9 - 15) + C = -6k + C \end{cases}\right\} \Rightarrow -2 = \frac{4}{3}k \Rightarrow k = -\frac{3}{2}$$

$\Downarrow$

$C = -6$

$f(x) = -\frac{3}{2} \cdot \left(\frac{1}{3}x^3 - 5x\right) - 6 \Rightarrow f(2) = -\frac{3}{2} \cdot \left(\frac{8}{3} - 10\right) - 6 = 5$

3b. Suppose that a continuous function g satisfies:

$$\int_0^3 g(x)\,dx = 7 \quad \text{and} \quad \int_0^6 g(2x)\,dx = 5$$

Find $\int_1^2 x\,g(3x^2)\,dx$.

$$5 = \int_0^6 g(2x)\,dx = \int_0^{12} g(u) \cdot \frac{1}{2}\,du \Rightarrow \int_0^{12} g(u)\,du = 10$$

$\boxed{u = 2x \quad du = 2\,dx}$

$$\int_1^2 x\,g(3x^2)\,dx = \int_3^{12} g(u) \cdot \frac{1}{6}\,du = \frac{1}{6}\left(\int_0^{12} g(u)\,du - \int_0^3 g(u)\,du\right)$$

$\boxed{u = 3x^2 \quad du = 6x\,dx}$

$$= \frac{1}{6} \cdot (10 - 7) = \frac{1}{2}$$