4a. Evaluate the integral $\displaystyle\int \frac{\tan x}{2\tan^2 x - 1}\,dx$

$$\int \frac{\tan x}{2\tan^2 x - 1}\,dx = \int \frac{\sin x \cos x}{2\sin^2 x - \cos^2 x}\,dx = \frac{1}{6}\int \frac{du}{u} = \frac{1}{6}\ln|u| + C$$

$$\boxed{\begin{aligned} u &= 2\sin^2 x - \cos^2 x \\ du &= (2\cdot 2\sin x\cdot\cos x - 2\cos x\cdot(-\sin x))\,dx \\ &= 6\sin x\cos x\,dx \end{aligned}}$$

$$= \frac{1}{6}\ln|2\sin^2 x - \cos^2 x| + C$$

4b. Suppose that a function f with a continuous derivative satisfies the conditions:

❶ $\displaystyle\int_1^3 f'(x)\,dx = 1$ ❷ $\displaystyle\int_1^3 f(x)\,f'(x)\,dx = 1$ ❸ $\displaystyle\int_1^3 x^2 f'(x)\,dx = 1$ ❹ $\displaystyle\int_1^3 (f(x))^2\,dx = 2020$

Find $\displaystyle\int_1^3 x f(x)\,dx$.

$$(1) \Rightarrow 1 = \int_1^3 f'(x)\,dx = f(x)\Big]_1^3 = f(3) - f(1)$$

$$(2) \Rightarrow 1 = \int_1^3 f(x)f'(x)\,dx = \frac{1}{2}f(x)^2\Big]_1^3 = \frac{1}{2}f(3)^2 - \frac{1}{2}f(1)^2$$

$$f(3) = \frac{3}{2}, \quad f(1) = \frac{1}{2}$$

$$(3) \Rightarrow 1 = \int_1^3 x^2 f'(x)\,dx = \int_1^3 x^2\,d(f(x)) = x^2 f(x)\Big]_1^3 - \int_1^3 f(x)\,d(x^2)$$

$$= 9f(3) - f(1) - 2\int_1^3 x f(x)\,dx = \frac{27}{2} - \frac{1}{2} - 2\int_1^3 x f(x)\,dx$$

$$= 13 - 2\int_1^3 x f(x)\,dx$$

$$\Rightarrow \int_1^3 x f(x)\,dx = 6$$