[*25 points*]

2. Find all pairs (a, b) of constants for which the limit

$$\lim_{x\to 0} \frac{e^x - xe^{ax} - \cos bx}{x^4}$$

exists, and find the value of the limit for each of these pairs of constants.

$$\lim_{x\to 0} \frac{e^x - xe^{ax} - \cos bx}{x^4} \overset{L'H}{=} \lim_{x\to 0} \frac{e^x - e^{ax} - axe^{ax} + b\sin bx}{4x^3}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{e^x - ae^{ax} - ae^{ax} - a^2xe^{ax} + b^2\cos bx}{12x^2}$$

L'H $\leftarrow$ $\boxed{\text{provided that } 1-2a+b^2=0\text{; otherwise, the limit does not exist}}$

$$= \lim_{x\to 0} \frac{e^x - 2a^2e^{ax} - a^2e^{ax} - a^3xe^{ax} - b^3\sin bx}{24x}$$

L'H $\leftarrow$ $\boxed{\text{provided that } 1-3a^2=0\text{; otherwise, the limit does not exist}}$

$$= \lim_{x\to 0} \frac{e^x - 3a^3e^{ax} - a^3e^{ax} - a^4xe^{ax} - b^4\cos bx}{24} = \frac{1-4a^3-b^4}{24}$$

$$1-3a^2=0 \implies a=\frac{1}{\sqrt{3}} \quad \text{or} \quad a=-\frac{1}{\sqrt{3}}$$

$$1-2a+b^2=0$$

$$b=\sqrt{\frac{2}{\sqrt{3}}-1} \quad \text{or} \quad b=-\sqrt{\frac{2}{\sqrt{3}}-1}$$

$b^2=-\frac{2}{\sqrt{3}}-1$, which is impossible

The limit exists exactly when $(a,b)=\left(\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{\sqrt{3}}-1}\right)$ or $\left(\frac{1}{\sqrt{3}}, -\sqrt{\frac{2}{\sqrt{3}}-1}\right)$, and in both cases the limit is

$$\frac{1-4a^3-b^4}{24} = \frac{1}{24}\left(1-\frac{4}{3\sqrt{3}}-\left(\frac{2}{\sqrt{3}}-1\right)^2\right) = \frac{1}{9\sqrt{3}}-\frac{1}{18}$$