**3a.** Suppose that a continuous function $f$ satisfies the equation

$$f'(x) = f(x)\int_0^x f(t)\,dt$$

for all $x$.

❶ In this part assume that $f(0) = -1$. Show that $f$ has a critical point at $x = 0$, and determine whether it is a local maximum, a local minimum, or neither.

$f'(0) = f(0)\int_0^0 f(t)\,dt = -1\cdot 0 = 0 \Rightarrow f$ has a critical point at $x=0$

$f''(x) = f'(x)\int_0^x f(t)\,dt + f(x)\cdot\left(\int_0^x f(t)\,dt\right)' \overset{\text{FTC1}}{=} f'(x)\int_0^x f(t)\,dt + f(x)^2$

$\Rightarrow f''(0) = f'(0)\cdot\int_0^0 f(t)\,dt + f(0)^2 = 0\cdot 0 + (-1)^2 = 1 > 0$

$\Rightarrow f$ has a local minimum at $x=0$

❷ In this part assume that $f''(2) = 0$. Express $f(2)$ in terms of $A = f'(2)$ only.

$f(x)f''(x) = f'(x)\cdot f(x)\int_0^x f(t)\,dt + f(x)^3 \Rightarrow f(x)f''(x) = f'(x)^2 + f(x)^3$

$\Rightarrow f(2)f''(2) = f'(2)^2 + f(2)^3 \Rightarrow 0 = A^2 + f(2)^3 \Rightarrow f(2) = -A^{2/3}$

**3b.** Suppose that a function $g$ with continuous second derivative on $[0,1]$ satisfies:

$$g(0) = 2,\quad g'(0) = 3,\quad g''(0) = 4,\quad g(1) = -4,\quad g'(1) = -3,\quad g''(1) = -2$$

Evaluate $\displaystyle\int_0^1 g(x)\,g'(x)\left(1 + g'(x)^2 + g(x)g''(x)\right)dx$ .

$\displaystyle\int_0^1 g(x)g'(x)\,dx = \int_{g(0)}^{g(1)} u\,du = \left.\frac{u^2}{2}\right]_2^{-4} = \frac{(-4)^2 - 2^2}{2} = 6$

$\boxed{\begin{array}{l} u = g(x) \\ du = g'(x)\,dx \end{array}}$ $\boxed{\begin{array}{l} u = g(x)g'(x) \\ du = (g'(x)^2 + g(x)g''(x))\,dx \end{array}}$

$\displaystyle\int_0^1 g(x)g'(x)\left(g'(x)^2 + g(x)g''(x)\right)dx = \int_{g(0)g'(0)}^{g(1)g'(1)} u\,du = \left.\frac{u^2}{2}\right]_{2\cdot 3}^{(-4)\cdot(-3)} = \frac{12^2 - 6^2}{2} = 54$

$\displaystyle\int_0^1 g(x)g'(x)\left(1 + g'(x)^2 + g(x)g''(x)\right)dx = 6 + 54 = 60$