4. For a positive continuous function f on $(-\infty,\infty)$, let

- $R(a)$ be the region between the graph of $y=f(x)$ and the x-axis for $x \le a$,
- $A(a)$ be the area of $R(a)$, and
- $V(a)$ be the volume of the solid generated by revolving $R(a)$ about the x-axis.

a. Show that if $f(x)=e^{2x/\pi}$, then $V(a)=(A(a))^2$ for all a.

$$A(a)=\int_{-\infty}^{a} f(x)\,dx=\int_{-\infty}^{a} e^{2x/\pi}\,dx=\frac{\pi}{2}e^{2a/\pi}$$

$$V(a)=\pi\int_{-\infty}^{a} f(x)^2\,dx=\pi\int_{-\infty}^{a} e^{4x/\pi}\,dx=\frac{\pi^2}{4}e^{4a/\pi}$$

$\Rightarrow V(a)=A(a)^2$ for all a

because for a positive constant k:

$$\int_{-\infty}^{a} e^{kx}\,dx=\lim_{c\to-\infty}\int_{c}^{a} e^{kx}\,dx=\lim_{c\to-\infty}\left[\frac{e^{kx}}{k}\right]_c^a=\lim_{c\to-\infty}\frac{e^{ka}-\overset{\nearrow 0}{e^{kc}}}{k}=\frac{e^{ka}}{k}$$

b. Show that $f(x)=e^{2x/\pi}$ is the only positive continuous function on $(-\infty,\infty)$ for which $A(a)$ and $V(a)$ are finite and satisfy $V(a)=(A(a))^2$ for all a, and $f(0)=1$.

$$V(a)=A(a)^2 \Rightarrow \pi\int_{-\infty}^{a} f(x)^2\,dx=\left(\int_{-\infty}^{a} f(x)\,dx\right)^2$$

$$\Rightarrow \pi\frac{d}{da}\int_{-\infty}^{a} f(x)^2\,dx=\frac{d}{da}\left(\int_{-\infty}^{a} f(x)\,dx\right)^2 \xrightarrow{\text{FTC1}} \pi f(a)^2=2\int_{-\infty}^{a} f(x)\,dx\cdot f(a)$$

$$\Rightarrow \pi f(a)=2\int_{-\infty}^{a} f(x)\,dx \xrightarrow{\text{FTC1}} \pi f'(a)=2f(a)$$

$$\Rightarrow \frac{f'(a)}{f(a)}=\frac{2}{\pi} \Rightarrow \int\frac{f'(a)}{f(a)}\,da=\int\frac{2}{\pi}\,da \Rightarrow \ln|f(a)|=\frac{2}{\pi}a+C$$

for all a

$\boxed{a=0}$

$$\Rightarrow 0=\ln|\underbrace{f(0)}_{1}|=C$$

Hence $f(a)=e^{2a/\pi}$ for all a as $f(a)>0$.