

1. Let $f(x) = x^5 - 3x^3 - x + 4$.

a. Compute $f'(x)$.

$$f'(x) = 5x^4 - 9x^2 - 1$$

b. Find the x -coordinates of all points on the graph of $y = f(x)$ where the tangent line is horizontal.

$$f'(x) = 0 \Rightarrow 5x^4 - 9x^2 - 1 = 0 \Rightarrow x^2 = \frac{9 \pm \sqrt{101}}{10}$$

$$\Rightarrow x^2 = \frac{9 + \sqrt{101}}{10} \Rightarrow x = \pm \sqrt{\frac{9 + \sqrt{101}}{10}}$$

$x^2 \geq 0$

c. Show that the equation $f(x) = 0$ has at most three real solutions.

Suppose $r_1 < r_2 < r_3 < r_4$ satisfy $f(r_1) = f(r_2) = f(r_3) = f(r_4) = 0$.

Since f is continuous and differentiable everywhere, there exist $r_1 < c_1 < r_2 < c_2 < r_3 < c_3 < r_4$ such that $f'(c_1) = f'(c_2) = f'(c_3) = 0$ by Rolle's Theorem. This contradicts the fact that f' has only two zeros. Therefore f cannot have more than three zeros.

d. Show that the equation $f(x) = 0$ has at least three real solutions.

$$f(-2) = -2 < 0$$

$$f(0) = 4 > 0$$

$$f(3/2) = -11/32 < 0$$

$$f(2) = 10 > 0$$

Since f is continuous everywhere, applying the Intermediate Value Theorem to the intervals $[-2, 0]$, $[0, 3/2]$ and $[3/2, 2]$, we conclude that f has a zero in each of the intervals $(-2, 0)$, $(0, 3/2)$ and $(3/2, 2)$.