2. A snowman is an anthropomorphic sculpture made from snow as well as some pieces of coal, a carrot, a hat and a scarf. For the purposes of this question, we consider a snowman to consist of a spherical head of radius a and a spherical body of radius b, and we also assume that the snow does not melt and its density does not change while it is being sculpted.

The research shows that the *cuteness* $\mathcal{K}$ of a snowman is given by

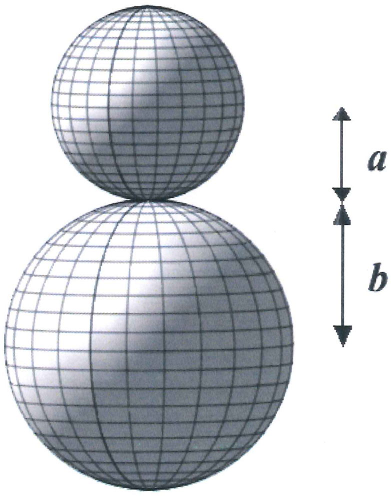
$$\mathcal{K} = \begin{cases} \left(\frac{a}{b}\right)^2 \left(1 - \frac{a}{b}\right)(a^2 + ab + b^2) & \text{if } 0 \le a < b, \\ 0 & \text{if } 0 < b \le a. \end{cases}$$

Find the dimensions of the cutest snowman that can be built with $4\pi/3$ m^3 of snow.

$$\frac{4\pi}{3}a^3 + \frac{4\pi}{3}b^3 = \text{Total volume} = \frac{4\pi}{3}\ \text{m}^3$$

$$\Rightarrow b^3 = 1 - a^3$$

$a \le b$



$$K = \frac{a^2}{b^3}(b^3 - a^3) = \frac{a^2(1-2a^3)}{1-a^3} \quad \text{for } 0 \le a \le \frac{1}{\sqrt[3]{2}},$$

0 otherwise.

$$\boxed{\text{Maximize } K = \frac{a^2 - 2a^5}{1-a^3} \text{ for } 0 \le a \le \frac{1}{\sqrt[3]{2}}.}$$

Critical points: $\dfrac{dK}{da} = \dfrac{(2a-10a^4)\cdot(1-a^3) - (-3a^2)(a^2-2a^5)}{(1-a^3)^2} = 0$

$\Rightarrow 4a^7 - 9a^4 + 2a = 0 \Rightarrow a(a^3-2)(4a^3-1) = 0 \Rightarrow a = 0,\ a = \sqrt[3]{2},\ a = \dfrac{1}{\sqrt[3]{4}}$

endpoint $\quad \sqrt[3]{2} > 1/\sqrt[3]{2}$

$K(1/\sqrt[3]{4}) = \dfrac{1}{3\cdot\sqrt[3]{2}} > 0$

Largest

Endpoints:

$K(0) = 0$

$K(1/\sqrt[3]{2}) = 0$

The maximum cuteness occurs for $a = \dfrac{1}{\sqrt[3]{4}}$ m and $b = \sqrt[3]{\dfrac{3}{4}}$ m.