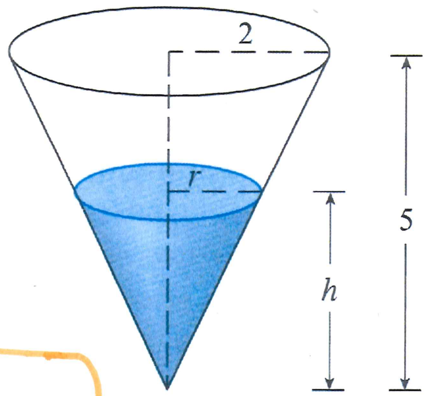
4. A water tank has the shape of an upside-down cone with radius 2 m and height 5 m. The water is running out of the tank through a small hole at the bottom. Assume that the speed of the water flowing through the hole is proportional to the square root of the depth of the water in the tank. ⊛



a. In this part, suppose that the water is running out at a rate of 3 m^3/min when the depth of the water in the tank is 4 m. Find the rate at which the water level is changing at this moment.

$$\frac{r}{h} = \frac{2}{5}$$

$$V = \text{Volume of water} = \frac{\pi}{3} r^2 h = \frac{\pi}{3}\left(\frac{2}{5}h\right)^2 h = \frac{4\pi}{75} h^3$$

$$\Downarrow d/dt$$

$$\frac{dV}{dt} = \frac{4\pi}{25} h^2 \frac{dh}{dt}$$

$$\Downarrow \text{at our moment}$$

$$-3\ m^3/min = \frac{4\pi}{25} \cdot (4m)^2 \cdot \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = -\frac{75}{64\pi}\ m/min$$

The water level is falling at a rate of $\frac{75}{64\pi}$ m/min.

b. In this part, suppose that the water level is falling at a rate of 1/3 m/min when the tank is full. Find the rate at which the water level is changing when the depth of the water in the tank is 4 m.

$$⊛ \Rightarrow \frac{dV}{dt} = -k\sqrt{h} \quad \text{for some positive constant } k$$

$$\Downarrow$$

$$\frac{4\pi}{25} h^2 \frac{dh}{dt} = -k\sqrt{h}$$

when $h = 5m$: $\frac{4\pi}{25} \cdot 5^2 \cdot \left(-\frac{1}{3}\right) = -k\sqrt{5}$

when $h = 4m$: $\frac{4\pi}{25} \cdot 4^2 \cdot \frac{dh}{dt} = -k\sqrt{4}$

$$\frac{dh}{dt} = -\frac{5\sqrt{5}}{24}\ m/min$$

The water level is falling at a rate of $\frac{5\sqrt{5}}{24}$ m/min.