Induction and Sequences

Let n_0 be an integer. Suppose that we want to prove that a statement $S(n)$ about integers $n \geq n_0$ is true for all $n \geq n_0$. One way of doing this is to use the Induction Method. If we can prove that

- $S(n_0)$ is true, and
- For each $k \geq n_0$, if $S(k)$ is true then $S(k+1)$ is true,

then it follows that $S(n)$ is true for all $n \geq n_0$.

Example 0: Consider the sequence $a_n = \dfrac{10^n}{n!}$ for $n \geq 0$. Then

$$a_0 = 1,\ a_1 = 10,\ a_2 = 50,\ a_3 = \frac{500}{3},\ a_4 = \frac{1250}{3}\ .$$

In particular, $a_0 < a_1 < a_2 < a_3 < a_4$. Can we conclude that $\{a_n\}_{n=0}^{\infty}$ is an increasing sequence? No, because, for instance,

$$a_{10} = \frac{15262500}{567} > \frac{15625000}{6237} = a_{11}\ .$$

In fact, we can prove that the sequence $\{a_n\}_{n=10}^{\infty}$ is decreasing. If $n \geq 10$, then

$$a_{n+1} = \frac{10^{n+1}}{(n+1)!} = \frac{10^n}{n!} \cdot \frac{10}{n+1} \leq \frac{10^n}{n!} \cdot \frac{10}{11} < \frac{10^n}{n!} = a_n\ .$$

Example 1: Consider the sequence recursively defined by the conditions

$$a_1 = 7 \ \text{ and } \ a_n = \frac{a_{n-1}+1}{4} \ \text{ for } \ n \geq 2\,.$$

We want to show that this sequence is convergent using the Monotonic Sequence Theorem. First we will show that the sequence is bounded below by 0.

Claim: $a_n > 0$ for all $n \geq 1$.

Proof: We will use induction.

- For $n = 1$, we have $a_1 = 7 > 0$.
- Suppose that $a_k > 0$ for some $k \geq 1$. Then

$$a_k > 0 \Longrightarrow a_k + 1 > 1 \Longrightarrow a_{k+1} = \frac{a_k+1}{4} > \frac{1}{4} > 0$$

and we are done. □

Next we will show that the sequence is decreasing.

Claim: $a_{n+1} < a_n$ for all $n \geq 1$.

Proof: We will use induction again.

- For $n = 1$, we have $a_2 = \dfrac{7+1}{4} = 2 < 7 = a_1$.
- Suppose that $a_{k+1} < a_k$ for some $k \geq 1$. We want to show that $a_{k+2} < a_{k+1}$. This is true, because

$$a_{k+1} < a_k \Longrightarrow a_{k+1} + 1 < a_k + 1 \Longrightarrow a_{k+2} = \frac{a_{k+1}+1}{4} < \frac{a_k+1}{4} = a_{k+1}\ .$$ □

Finally, since the sequences is bounded from below and decreasing, it is convergent by the Monotonic Sequence Theorem.

Example 2: In this Example we will again show that the sequence defined by

$$a_1 = 7 \text{ and } a_n = \frac{a_{n-1}+1}{4} \text{ for } n \geq 2$$

is convergent. But this time we will not use induction in the second part of the proof.

First we will show that the sequence is bounded below by 1/3.

Claim: $a_n > 1/3$ for all $n \geq 1$.

Proof: We will use induction.

- For $n = 1$, we have $a_1 = 7 > 1/3$.
- Suppose that $a_k > 1/3$ for some $k \geq 1$. Then:

$$a_k > \frac{1}{3} \Longrightarrow a_k + 1 > \frac{4}{3} \Longrightarrow a_{k+1} = \frac{a_k+1}{4} > \frac{1}{3} \qquad \square$$

Now we will prove that the sequence is decreasing. As we already now that $a_n > 1/3$ and therefore $1 - 3a_n < 0$ for all $n \geq 1$, it immediately follows that

$$a_{n+1} - a_n = \frac{a_n+1}{4} - a_n = \frac{1}{4}(1-3a_n) < 0$$

and hence $a_{n+1} < a_n$ for all $n \geq 1$.

Once again, since the sequences is bounded from below and decreasing, it is convergent by the Monotonic Sequence Theorem.

Example 3: In this Example we will one more time show that the sequence defined by

$$a_1 = 7 \text{ and } a_n = \frac{a_{n-1}+1}{4} \text{ for } n \geq 2$$

is convergent, but this time without using the Monotonic Sequence Theorem.

Note that

$$a_{n+1} - \frac{1}{3} = \frac{a_n+1}{4} - \frac{1}{3} = \frac{1}{4}\left(a_n - \frac{1}{3}\right)$$

for $n \geq 1$. Therefore, we obtain

$$a_n - \frac{1}{3} = \frac{1}{4}\left(a_{n-1} - \frac{1}{3}\right) = \left(\frac{1}{4}\right)^2\left(a_{n-2} - \frac{1}{3}\right) = \cdots = \left(\frac{1}{4}\right)^{n-2}\left(a_2 - \frac{1}{3}\right) = \left(\frac{1}{4}\right)^{n-1}\left(a_1 - \frac{1}{3}\right)$$

and use this to conclude that

$$\lim_{n\to\infty}\left(a_n - \frac{1}{3}\right) = \lim_{n\to\infty}\left(\left(\frac{1}{4}\right)^{n-1}\left(a_1 - \frac{1}{3}\right)\right) = \left(a_1 - \frac{1}{3}\right)\lim_{n\to\infty}\left(\frac{1}{4}\right)^{n-1} = 0$$

as $\lim_{n\to\infty} r^n = 0$ for $|r| < 1$. Hence $\lim_{n\to\infty} a_n = \frac{1}{3}$.

Two remarks about this proof: Firstly, note that this time we did not use the fact $a_1 = 7$ at all. That is, we proved that *any* sequence defined by this recursion relation converges to 1/3, no matter what the initial condition is. Secondly, there is an induction hidden somewhere in there. Find it and write it out explicitly as an exercise.