Bilkent Calculus II Exams
1988-2012
Version 7.2

Spring 2012 Midterm I

1a. Write in the box an equation for one of the planes that contain the line $x = -2t + 3, y = 5t + 1, z = 4t - 1, -\infty < t < \infty$. No explanation is required.

1b. Write in the box parametric equations for one of the lines that are contained in the plane $7x - y - 2z = 11$. No explanation is required.

1c. Find an equation of the plane that passes through the point $P(-3, 2, 1)$ and is perpendicular to both of the planes with equations $x+3y-8z = 2$ and $2x-y+6z = 1$.

2a. Find the length of the parametric curve $x = (1+\cos t)\cos t, y = (1+\cos t)\sin t$, $0 \le t \le 2\pi$.

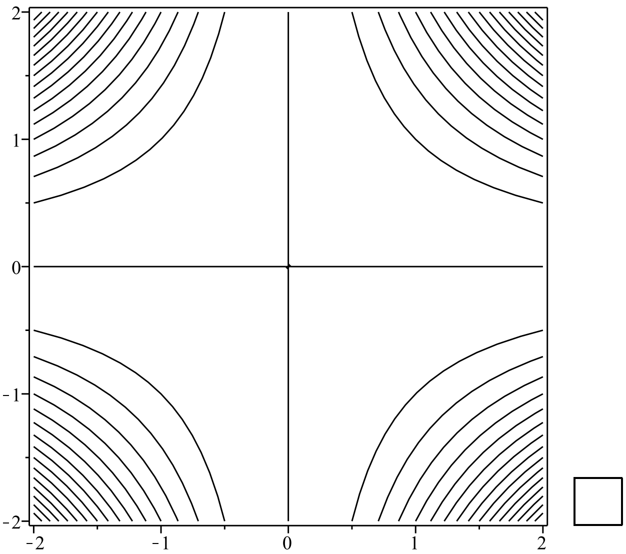
2b. Find parametric equations of the tangent line to the parametric curve $\mathbf{r} = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$, $-\infty < t < \infty$, at the point with $t = 2$.

3. The level curves of the following five functions are shown in the figures below. Match these with their functions by filling in the boxes with the corresponding letters.

A. $f(x, y) = \sin x + 2\sin y$ **B.** $f(x, y) = (4x^2 + y^2)e^{-x^2-y^2}$ **C.** $f(x, y) = x^2y^2$

D. $f(x, y) = xye^{-y^2}$ **E.** $f(x, y) = 3x - 2y$

(*The figures are on the next page.*)



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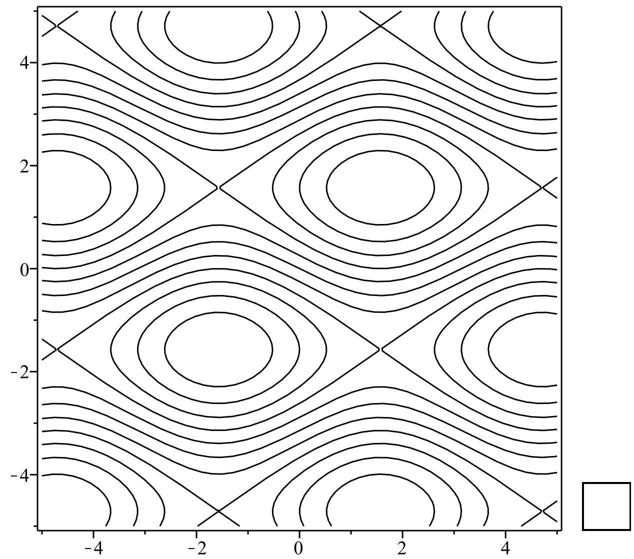

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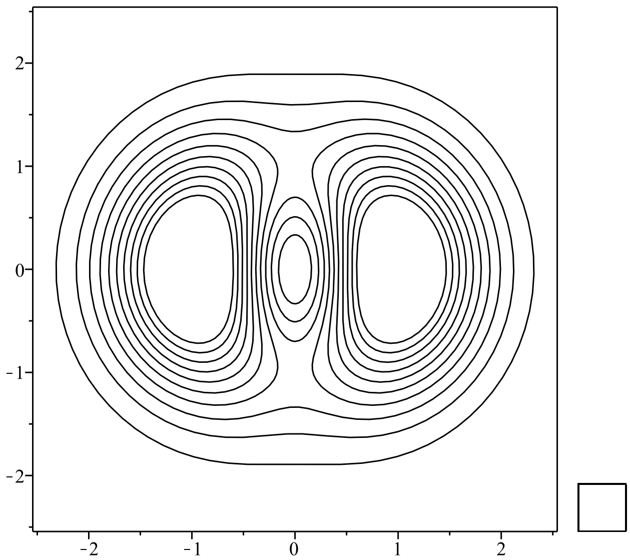

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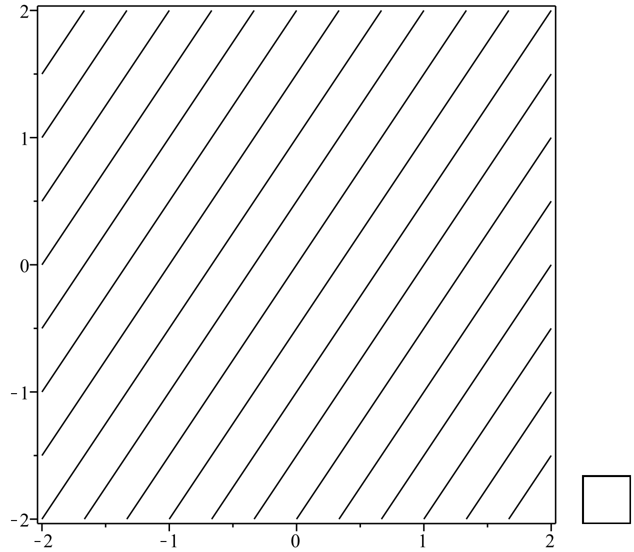

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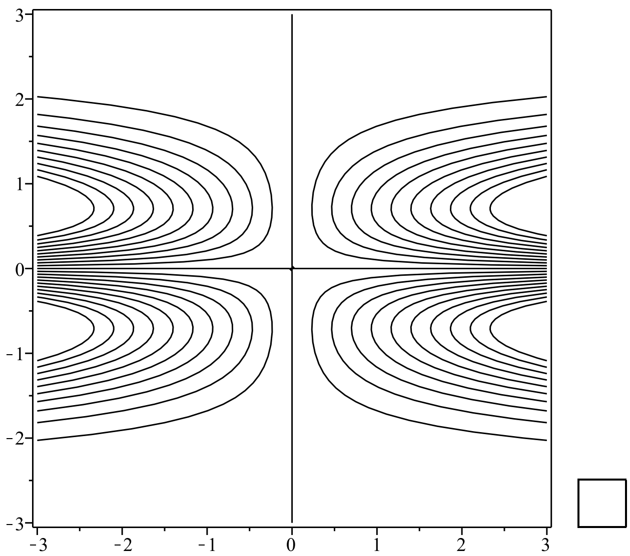


-3 -2 -1 0 1 2 3

4. Let $f(x,y) = \dfrac{x^3y^4}{x^4 + x^3y^2 + y^{10}}$.

a. Show that the limit of $f(x,y)$ as (x,y) approaches $(0,0)$ along any line through the origin is the same.

b. Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

5. Let $u = x + y + z$, $v = xy + yz + zx$, $w = xyz$, and suppose that $f(u,v,w)$ is a differentiable function satisfying $f(u,v,w) = x^4 + y^4 + z^4$ for all (x,y,z). Find $f_u(2,-1,-2)$.

Spring 2012 Midterm II

1a. In **(i-iii)**, if there is a differentiable function $f(x,y)$ whose derivatives at $(0,0)$ in the directions of the vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are all positive, give an example of such a function; if there is no such function, write DOES NOT EXIST in the box. No explanation is required.

i. $\mathbf{A} = \mathbf{i} + 2\mathbf{j}$, $\mathbf{B} = \mathbf{i} - \mathbf{j}$, $\mathbf{C} = \mathbf{i}$

$f(x,y) =$ [ ]

ii. $\mathbf{A} = \mathbf{i} + 2\mathbf{j}$, $\mathbf{B} = \mathbf{i} - \mathbf{j}$, $\mathbf{C} = -\mathbf{i} - \mathbf{j}$

$f(x,y) =$ [ ]

iii. $\mathbf{A} = 3\mathbf{i} + \mathbf{j}$, $\mathbf{B} = \mathbf{i} - \mathbf{j}$, $\mathbf{C} = -\mathbf{i} - \mathbf{j}$

$f(x,y) =$ [ ]

1b. A bug is standing on the ground at a point P. If it moves towards north from P, the temperature decreases at a rate of 4 C°/m. If it moves towards southeast from P, the temperature increases at a rate of $3\sqrt{2}$ C°/m. In which direction should the bug move to go to cooler points as fast as possible? (Choose a coordinate system on the ground with the positive x-axis pointing east and the positive y-axis pointing north, and express your answer as a unit vector.)

2. Find the absolute maximum value of $f(x,y) = x^3 - xy - y^2 + 2y$ on the square $R = \{(x,y) : 0 \le x \le 1 \text{ and } 0 \le y \le 1\}$.

3. Find and classify the critical points of $f(x, y) = x^3 - 2x^2 + xy^2$.

4. Evaluate the following integrals.

a. $\displaystyle\iint_R |\cos(x+y)|\, dA$ where $R = \{(x, y) : 0 \le x \le \pi/2 \text{ and } 0 \le y \le \pi/2\}$.

b. $\displaystyle\int_0^1 \int_0^{\sqrt{y-y^2}} \frac{dx\, dy}{(x^2+y^2-(x^2+y^2)^2)^{1/2}}$

5. Let D be the region in the first octant bounded by the coordinate planes, the plane $x + y = 4$, and the cylinder $y^2 + 4z^2 = 16$.

- Choose *two* of the following rectangular boxes by putting a ✗ in the small square in front of them, and then
- choose *one* of the orders of integration in each of the selected boxes by putting a ✗ in the small square in front of them.

☐ [☐ $dx\,dy\,dz$ ☐ $dx\,dz\,dy$]	☐ [☐ $dy\,dx\,dz$ ☐ $dy\,dz\,dx$]	☐ [☐ $dz\,dx\,dy$ ☐ $dz\,dy\,dx$]

Express the volume V of the region D as iterated integrals in both of your selected orders of integration **(a)** and **(b)**. (*Do not evaluate the integrals!*)

Spring 2011 Midterm I

1. Consider the power series $\displaystyle\sum_{n=1}^{\infty} \frac{x^{n^2}}{2^n n}$.

a. Write the first three nonzero terms of the power series.

b. Find the radius of convergence of the power series.

c. Find the exact value of the sum of the power series at $x = -1$.

2a. Find the coefficient of x^{2011} in the Maclaurin series of $f(x) = xe^{-x^2}$.

2b. Exactly one of the following statements is true. Choose the true statement and mark the box in front of it with a ✓.

☐ If $\displaystyle\sum_{n=1}^{\infty} a_n^2$ converges, then $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

☐ If $\displaystyle\sum_{n=1}^{\infty} a_n^2$ converges, then $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} a_n^3$ converges.

Now either

- ☐ Prove the true statement, or
- ☐ Give an example that shows the other statement is not true.

Indicate the task you choose with a ✗.

3a. Find the equation of the plane containing the line $L_1 : x = 2t+3, y = 4t-1, z = -t+2$, $-\infty < t < \infty$, and parallel to the line $L_2 : x = 2s+3, y = s+2, z = 2s-2$, $-\infty < s < \infty$.

3b. When a wheel of unit radius rolls along the x-axis the path traced by a point P on its circumference is given by $\mathbf{r} = (t - \sin t)\,\mathbf{i} + (1 - \cos t)\,\mathbf{j}$, $-\infty < t < \infty$. Find the distance traveled by P during one full turn of the wheel.

4a. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^6+y^2} = 0$.

4b. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^6y^4}{(x^6+y^2)^3}$ does not exist.

5. Find all possible values of the constants a and b such that the function

$$f(x,y) = y^a e^{bx^2/y}$$

satisfies the equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{2}{x}\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$$

for all (x,y) with $x > 0$ and $y > 0$.

Spring 2011 Midterm II

1. Let $P_0(2,2,1)$ and suppose that $f(x,y,z)$ and $g(x,y,z)$ are differentiable functions satisfying the following conditions:

i. $f(P_0) = 1$ and $g(P_0) = 6$.

ii. $\left.\dfrac{\partial g}{\partial x}\right|_{P_0} = -1$.

iii. At P_0, f increases fastest in the direction of the vector $\mathbf{A} = 4\mathbf{i} - \mathbf{j} - 8\mathbf{k}$ and its derivative in this direction is 7.

iv. The tangent plane of the surface defined by the equation

$$f(x,y,z) + 2g(x,y,z) = 13$$

at the point P_0 has the equation $5x + y - z = 11$.

Find $\left.\dfrac{\partial g}{\partial z}\right|_{P_0}$.

2. Each of the following functions has a critical point at $(0,0)$. Indicate the type of this critical point by marking the corresponding box with a ✗. No explanation is required. No partial credit will be given.

a. $f(x,y) = x^3y^3$ has a

☐ local maximum

☐ local minimum

☐ saddle point

☐ none of the above

at $(0,0)$.

b. $f(x,y) = 1 - x^2y^2$ has a

☐ local maximum

☐ local minimum

☐ saddle point

☐ none of the above

at $(0,0)$.

c. $f(x,y) = y^2 - yx^2$ has a

☐ local maximum

☐ local minimum

☐ saddle point

☐ none of the above

at $(0,0)$.

d. $f(x,y) = x^2 - x^2y + y^2$ has a

☐ local maximum

☐ local minimum

☐ saddle point

☐ none of the above

at $(0,0)$.

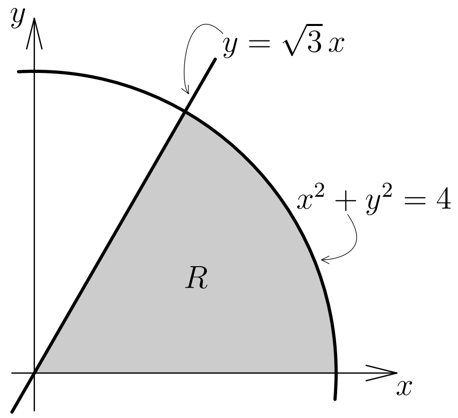
3. A flat circular plate has the shape of the region $x^2+y^2 \leq 1$. The plate, including the boundary where $x^2+y^2=1$, is heated so that the temperature at a point (x,y) is

$$T(x,y) = x^2 + 2y^2 - x.$$

Find the temperatures at the hottest and coldest points on the plate.

4. Evaluate the following integrals.

a. $\displaystyle\iint_R \sqrt{x^2+y^2}\,dA$ where R is the region shown in the figure.



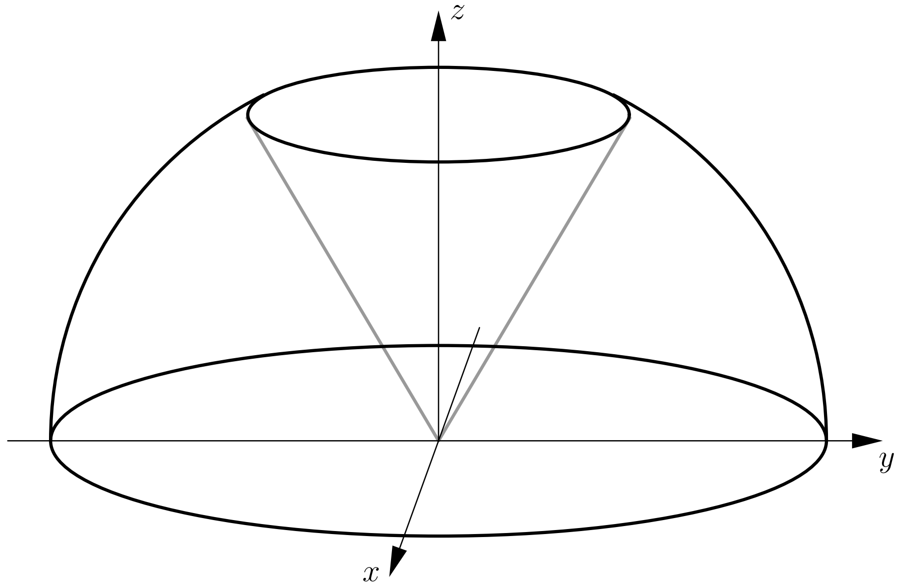
b. $\displaystyle\int_0^1 \int_1^{1/x^2} xy^2 e^{-y^2}\,dy\,dx$

5. Let D be the region in space lying inside the sphere $x^2+y^2+z^2=4$, outside the cone $z^2=3(x^2+y^2)$, and above the xy-plane. Fill in the boxes in parts **(a-c)** so that the right sides of the equalities become iterated integrals expressing the volume V of D in the given coordinates and orders of integration. No explanation is required.

a. $$V=\int_{\boxed{}}^{\boxed{}}\int_{\boxed{}}^{\boxed{}}\int_{\boxed{}}^{\boxed{}}\boxed{}\,d\rho\,d\phi\,d\theta$$

b. $$V=\int_{\boxed{}}^{\boxed{}}\int_{\boxed{}}^{\boxed{}}\int_{\boxed{}}^{\boxed{}}\boxed{}\,dz\,dr\,d\theta$$
$$+\int_{\boxed{}}^{\boxed{}}\int_{\boxed{}}^{\boxed{}}\int_{\boxed{}}^{\boxed{}}\boxed{}\,dz\,dr\,d\theta$$

c. $V = \displaystyle\int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} \square \; dr\,dz\,d\theta$



Spring 2011 Final

1a. Write the first three nonzero terms of the Maclaurin series of $\dfrac{x}{1+ax^2}$ where a is a constant.

1b. Write the first three nonzero terms of the Maclaurin series of $\sin(bx)$ where b is a constant.

1c. Find the constants a, b, c, d if $\dfrac{x}{1+ax^2} - \sin(bx) = x^3 + cx^4 + dx^5 + \cdots$ on some open interval containing $x = 0$.

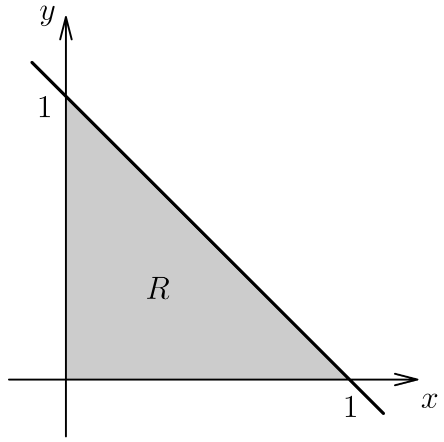
2. Let $u = x^2y^3$, $v = \sin(\pi x)$, and $z = f(u, v)$ where f is a function with continuous second order partial derivatives satisfying:

$$f(4,0) = 10 \qquad f_u(4,0) = 5 \qquad f_v(4,0) = 7$$
$$f_{uu}(4,0) = -2 \qquad f_{uv}(4,0) = -1 \qquad f_{vv}(4,0) = 3$$

a. Find $\left.\dfrac{\partial z}{\partial x}\right|_{(x,y)=(-2,1)}$.

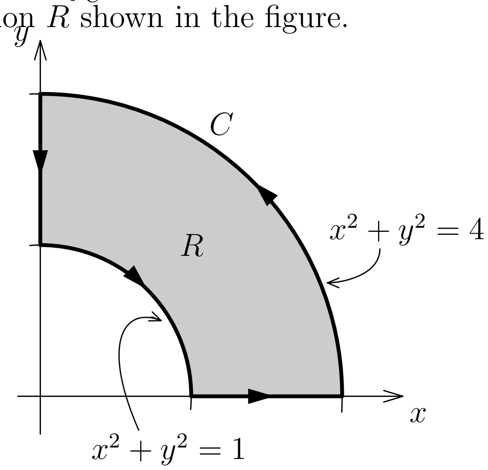
b. Find $\left.\dfrac{\partial^2 z}{\partial y \partial x}\right|_{(x,y)=(-2,1)}$.

3. Evaluate $\displaystyle\iint_R e^{(y-x)/(y+x)}\,dA$ where R is the region shown in the figure.



4a. Evaluate $\displaystyle\iiint_D \frac{dV}{\sqrt{x^2+y^2+(z-2)^2}}$ where D is the unit ball $x^2+y^2+z^2 \le 1$.

4b. Evaluate the line integral $\displaystyle\oint_C (6xy+\sin(x^2))\,dx+(5x^2+\sin(y^2))\,dy$ where C is the boundary of the region R shown in the figure.



5. Consider the parametrized surface $S : \mathbf{r} = u^2\mathbf{i} + \sqrt{2}\,uv\mathbf{j} + v^2\mathbf{k}$, $-\infty < u < \infty$, $0 \le v < \infty$. Find the area of the portion of the surface S that lies inside the unit ball $x^2+y^2+z^2 \le 1$.

Spring 2010 Midterm I

1a. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^6+y^2} = 0$.

1b. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^6+y^2}$ does not exist.

1c. Consider $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x\,|y|^a}{x^6+y^2}$ where a is a constant.

There is a real number A such that this limit is 0 if $a > A$, and this limit does not exist if $a < A$. What is A?

Write your answer here ☞ $A =$ ☐

No explanation is required and no partial points will be given in this part.

2. Assume that $f(x, y, z)$ is a differentiable function and at the point $P_0(1, -1, 2)$, f increases fastest in the direction of the vector $\mathbf{A} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

Exactly one of the following statements can be true about this function.

- Mark this statement with an ✓ and find $(\nabla f)_{P_0}$ assuming the statement to be true.
- Mark the other statement with an ✗ and explain why it cannot be true.

☐ The directional derivative of f at P_0 in the direction of the vector $\mathbf{B} = 2\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}$ is 5.

☐ The directional derivative of f at P_0 in the direction of the vector $\mathbf{B} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ is 5.

3. Find the points on the surface $xy + yz + zx - x - z^2 = 0$ where the tangent plane is parallel to the xy-plane.

4. Find all possible values of the constants C and k such that the function $f(x, y) = C(x^2 + y^2)^k$ satisfies the equation $f_{xx} + f_{yy} = f^3$ for all $(x, y) \neq (0, 0)$.

5. Find the absolute maximum and minimum values of the function $f(x, y) = 2x^3 + 2xy^2 - x - y^2$ on the unit disk $D = \{(x, y) : x^2 + y^2 \leq 1\}$.

Spring 2010 Midterm II

1. Evaluate the integral $\displaystyle\int_0^{\pi} \int_{x/\pi}^{1} y^4 \sin(xy^2)\, dy\, dx$.

2. Let D be the region in space bounded by the plane $y + z = 1$ on the top, the parabolic cylinder $y = x^2$ on the sides, and the xy-plane at the bottom.

- Choose *two* of the following rectangular boxes by putting a ✗ in the small square in front of them, and then
- choose *one* of the orders of integration in each of the selected boxes by putting a ✗ in the small square in front of them.

☐ (☐ $dx\,dy\,dz$ ☐ $dx\,dz\,dy$) ☐ (☐ $dy\,dx\,dz$ ☐ $dy\,dz\,dx$) ☐ (☐ $dz\,dx\,dy$ ☐ $dz\,dy\,dx$)

Express the volume V of the region D as iterated integrals in both of your selected orders of integration **(a)** and **(b)**. (*Do not evaluate the integrals!*)

3.

$$V = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_{x^2+y^2}^{\sqrt{12-x^2-y^2}} dz\,dy\,dx$$

expresses the volume V of a region D in space as an iterated integral in Cartesian coordinates.

Fill in the boxes in **(a)** and **(b)** so that the right sides of the equalities become iterated integrals expressing the volume of D in cylindrical and spherical coordinates, respectively. No explanation is necessary in this question. (*Do not evaluate the integrals!*)

a. $V = \int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} \square \; dz\,dr\,d\theta$

b. $V = \int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} \square \; d\rho\,d\phi\,d\theta$

$+ \int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} \square \; d\rho\,d\phi\,d\theta$

4. Find the value of the line integral

$$\oint_C (3x^2y^2 + y)\,dx + 2x^3y\,dy$$

where C is the cardioid $r = 1 + \cos\theta$ parameterized counterclockwise.

5. Let D be the closed ball $x^2 + y^2 + z^2 \leq 16$.

a. Show that for any vector field $\mathbf{F}$ that has continuous partial derivatives and satisfies the condition $|\mathbf{F}| \leq 5$ on D,

$$\iiint_D \nabla \cdot \mathbf{F}\, dV \leq 320\pi.$$

b. Give an example of a vector field $\mathbf{F}$ that has continuous partial derivatives and satisfies the condition $|\mathbf{F}| \leq 5$ on D such that

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = 320\pi,$$

and verify that the equality holds.

Spring 2010 Final

1. Let T be a transformation from the uv-plane to the xy-plane given by $x = f(u, v)$ and $y = g(u, v)$ where f and g are functions with continuous partial derivatives. Assume that T satisfies the condition that

$$(\text{Area of } T(G)) = \iint_G (u^2 + v^2)\, du\, dv$$

for every closed subset of the uv-plane on which T is one-to-one.

a. Let $G = \{(u, v) : 1 \leq u^2 + v^2 \leq 4\,, u \geq 0 \text{ and } v \geq 0\}$. Find the area of the image of G under T assuming that T is one-to-one on G.

b. If $f(u, v) = uv$, find a $g(u, v)$ that satisfies the conditions of the question. (You do not have to explain how you found $g(u, v)$, but you must verify that the conditions are satisfied.)

2. Determine whether each of the following series is convergent or divergent.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(\ln 2)^n}$

b. $\displaystyle\sum_{n=2}^{\infty} \frac{(\ln n)^3}{n^2}$

c. $\displaystyle\sum_{n=0}^{\infty} \frac{(5n)!}{30^n (2n)!(3n)!}$

3. Determine whether each of the following series is convergent or divergent.

a. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \cos\left(\frac{\pi}{n}\right)$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}\sin n}$

c. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n-2^n}{7^n-6^n}$

4. Consider the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{9n^2-1}$.

a. Find the radius of convergence of the power series.

b. Determine whether the power series converges or diverges at the **right** endpoint of its interval of convergence. If it converges, determine the type of convergence.

c. Determine whether the power series converges or diverges at the **left** endpoint of its interval of convergence. If it converges, determine the type of convergence.

5. In parts **(a-b)** of this question, if the series converges, write the exact sum of the series inside the box; and if the series diverges, write DIVERGES inside the box. No explanation is required. No partial points will be given.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{4^n(2n+1)} =$ ☐

b. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1} =$ ☐

In parts **(c-d)** of this question, if there exists a sequence $\{a_n\}$ satisfying the given conditions, write its general term inside the box; and if no such sequence exists, write DOES NOT EXIST inside the box. No explanation is required. No partial points will be given.

c. $1 \le a_n < a_{n+1}$ for all $n \ge 1$ and $\displaystyle\lim_{n\to\infty} a_n \ne \infty$.

$a_n =$ ☐

d. $\lim_{n\to\infty} na_n = 0$ and $\sum_{n=2}^{\infty} a_n$ diverges.

$a_n = \boxed{}$

Spring 2009 Midterm I

1. Determine whether each of the following series is convergent or divergent.

a. $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{2^n}{3^n}$

b. $\sum_{n=1}^{\infty}\frac{1}{1+(\ln n)^2}$

c. $\sum_{n=1}^{\infty}\cos(1/n)$

2. Determine whether each of the following series is convergent or divergent.

a. $\sum_{n=0}^{\infty}\frac{n!(n+1)!}{(2n+1)!}$

b. $\sum_{n=0}^{\infty}10^{-n^2/(n+1)}$

c. $\sum_{n=1}^{\infty}(e^{1/n^2}-1)$

3. Consider the power series $\sum_{n=0}^{\infty}\frac{x^n}{(n^2+1)(2^n+1)}$.

a. Find the radius of convergence of the power series.

b. Determine whether the power series is absolutely convergent, conditionally convergent or divergent at the **right** endpoint of its interval of convergence.

c. Determine whether the power series is absolutely convergent, conditionally convergent or divergent at the **left** endpoint of its interval of convergence.

4. Find the coefficient of the first nonzero term in the Maclaurin series generated by $f(x) = \sin x - \dfrac{x}{1 + x^2/6}$.

5a. Determine whether the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-4)^n}{n!(n+1)!}$ is positive or negative.

5b. Find the sum of the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{2^n(n^2-1)}$.

Spring 2009 Midterm II

1a. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{x^2+y^4} = 0$.

1b. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{(x^2+y^4)^3}$ does not exist.

2. A differentiable function $f(x, y, z)$ increases fastest at the point $P_0(2, 5, -1)$ in the direction of $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and at a rate of 7.

a. Find $\left.\dfrac{\partial f}{\partial z}\right|_{P_0}$.

b. Find the equation of the tangent plane to the level surface of f passing through the point P_0 .

3a. Show that if $f(z)$ is a differentiable function and $u(x, y) = f(x^2 - y^2)$, then $yu_x + xu_y = 0$ for all (x, y).

3b. Show that the parametric curve $\mathbf{r}(t) = e^t \cos t\,\mathbf{i} + e^t \sin t\,\mathbf{j}$, $-\infty < t < \infty$, cuts every circle with center at the origin at the same angle, and find this angle.

4. Find and classify the critical points of $f(x, y) = x^3 - 6x + x^2y^2$.

5. Evaluate the following integrals

a. $\displaystyle\int_0^{\pi^{1/6}} \int_{x^2}^{\pi^{1/3}} x^3 \sin(y^3)\, dy\, dx$

b. $\displaystyle\iint_R x^2 \sin(x^2 + y^2)\, dA$ where $R = \{(x, y) : x^2 + y^2 \leq \pi\}$.

Spring 2009 Final

1. In each of the following indicate whether the given series converges or diverges, and also indicate the *best* way of determining this by marking the corresponding boxes and filling in the corresponding blanks.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ ☐ converges ☐ diverges

☐ nth Term Test ☐ Direct Comparison Test with $\sum$ ________

☐ Ratio Test ☐ Limit Comparison Test with $\sum$ ________

☐ nth Root Test ☐ Alternating Series Test

b. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^5-1}}$ ☐ converges ☐ diverges

☐ nth Term Test ☐ Direct Comparison Test with $\sum$ ________

☐ Ratio Test ☐ Limit Comparison Test with $\sum$ ________

☐ nth Root Test ☐ Alternating Series Test

c. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{\sqrt{2}}}{2^{\sqrt{n}}}$ ☐ converges ☐ diverges

☐ nth Term Test ☐ Direct Comparison Test with $\sum$ ________

☐ Ratio Test ☐ Limit Comparison Test with $\sum$ ________

☐ nth Root Test ☐ Alternating Series Test

d. $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{n}{\ln n}$ ☐ converges ☐ diverges

☐ nth Term Test ☐ Direct Comparison Test with $\sum$ ________

☐ Ratio Test ☐ Limit Comparison Test with $\sum$ ________

☐ nth Root Test ☐ Alternating Series Test

e. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n-2^n}{\pi^n-3^n}$ ☐ converges ☐ diverges

☐ nth Term Test ☐ Direct Comparison Test with $\sum$ ________

☐ Ratio Test ☐ Limit Comparison Test with $\sum$ ________

☐ nth Root Test ☐ Alternating Series Test

2. Find the closest and farthest points on the sphere $x^2 + y^2 + z^2 = 4$ to the point $P(3, 1, -1)$.

3a. Evaluate the integral $\displaystyle\iiint_D \frac{z^2}{(x^2 + y^2 + z^2)^3}\, dV$ where $D = \{(x, y, z) : x^2 + y^2 \geq 1\}$.

3b. Express the iterated integral in cylindrical coordinates $\displaystyle\int_0^{\pi}\int_0^1\int_0^{\sqrt{4-r^2}} dz\, dr\, d\theta$ in terms of iterated integrals in spherical coordinates. Do not evaluate.

4a. Find a function $f(x, y)$ such that, for any region G in the first quadrant of the xy-plane, the double integral $\displaystyle\iint_G f(x, y)\, dx\, dy$ gives the area of $T(G)$ where T is the transformation $u = x^2/y$, $v = x/y^2$.

4b. Evaluate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = 3x^2y^2z\,\mathbf{i} + 2(x^3z + z^2)y\,\mathbf{j} + (x^3 + 2z)y^2\,\mathbf{k}$ and $C : \mathbf{r} = (t^3 - 2t)\,\mathbf{i} + (t^4 - 4t^2 - 1)\,\mathbf{j} + \cos(\pi t)\,\mathbf{k}$, $0 \leq t \leq 1$.

5. Let C be the unit circle in the xy-plane, and S be the boundary of the square $K = \{(x, y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$. If a is a constant such that

$$\oint_C (y^3 + xy^2 + xy)\, dx + (3xy^2 + x^2y + ax)\, dy = 1 ,$$

evaluate

$$\oint_S (y^3 + xy^2 + xy)\, dx + (3xy^2 + x^2y + ax)\, dy .$$

Spring 2008 Midterm I

1. a. Evaluate $\displaystyle\lim_{n\to\infty} \frac{(n!)^2(3n)^n}{n^n(2n)!}$.

b. Does the series $\displaystyle\sum_{n=2}^{\infty} \left(\frac{\sin n}{n^2} + \frac{1}{n(\ln n)^2}\right)$ converge or diverge? Justify!

2. a. Find the Taylor series generated by $f(x) = (1 + x^2)^{1/3}$ at $x = 0$.

b. Use the series in part **(a)** to estimate $\int_0^{1/2} (1+x^2)^{1/3}$ with error less than 0.01.

3. Find the radius and the interval of convergence of the series $\sum_{n=1}^{\infty} (-1)^n \frac{(x+2)^n}{\sqrt{n(n+1)}}$.

4. a. Find parametrization for the line in which the planes $5x - 2y = 11$ and $4y - 5z = -17$ intersect.

b. Write $\vec{u} = \mathbf{j} + \mathbf{k}$ as the sum of a vector parallel to $\vec{v} = \mathbf{i} + \mathbf{j}$ and a vector orthogonal to $\vec{v}$.

5. a. Let $\vec{r}_1(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ and $\vec{r}_2(t) = \sin s\,\mathbf{i} + \cos s\,\mathbf{k}$ be two curves. Find the intersection points of $\vec{r}_1$ and $\vec{r}_2$. Moreover, find the angle between the tangent vectors at each intersection point.

b. Find the point of intersection of the lines $x = 2t+1$, $y = 3t+2$, $z = 4t+3$ and $x = s+2$, $y = 2s+4$, $z = -4s-1$. Then find the plane determined by these lines.

Spring 2008 Midterm II

1. Let D be the solid bounded by the surfaces $x^2+y^2+z^2 = 9$ and $x^2+y^2+z^2 = 1$, and lying above the surface $z^2 = 3(x^2+y^2)$ with $z \geq 0$. Write down three integrals in rectangular, cylindrical, and spherical coordinates that give the volume of the solid. Do not evaluate these integrals.

2. a. Evaluate $\int_0^9 \int_{\sqrt{y}}^3 \sin(x^3)\, dx\, dy$.

b. Evaluate $\int_0^2 \int_0^{\sqrt{1-(x-1)^2}} \frac{x+y}{x^2+y^2}\, dy\, dx$.

3. Find the shortest distance from the origin to the surface $xyz^2 = 2$. (Explain why it is the shortest and not the longest.)

4. a. Let $f(x,y) = x^2y - 2xy + y^2 - 15y$. Find and classify the critical points of f.

b. Find the direction of most rapid increase for f at the point $(1,1)$ and the rate of change of f in this direction.

5. a. Let $z = f(x, y)$ be a differentiable function of two independent variables x and y such that $f(2, 1) = 3$, $f_x(2, 1) = 2$, $f_y(2, 1) = -1$. Define another function $z = g(x, y)$ of two independent variables x and y as follows:

$$g(x, y) = f\left(\frac{2}{x^2 + y^2}, \frac{y}{x}\, e^{xy}\right)$$

Find the equation of the tangent plane to the surface $z = g(x, y)$ at the point $(x, y) = (1, 0)$.

b. Using the tangent plane in part (**a**) approximate $g(1.1, -0.2)$.

Spring 2008 Final

1. Find the radius and the interval of convergence of the series

$$\sum_{n=2}^{\infty} \frac{(-2)^n (n + 1)}{(3n^2 + 1)\ \ln n} (x - 5)^n .$$

Explain your reasoning and be sure to check the endpoints.

2. Find the absolute maximum and minimum values of the function $f(x, y) = yx^2 - y^2 + 4$ on the unit disk $D = \{(x, y) : x^2 + y^2 \leq 1\}$.

3. a. Find the work done by the vector field

$$\mathbf{F}(x, y, z) = (2xz + 2)\mathbf{i} + (2y + z)j + (x^2 + y)\mathbf{k}$$

over the curve $\mathbf{r}(t) = (\sin t + 2\cos t - 2)\mathbf{i} + e^{t^2 - \pi t}\mathbf{j} + (t/\pi - 5)\mathbf{k}$ for t in $[0, \pi]$. (Hint: First try to answer the question "Is $\mathbf{F}$ a conservative field?")

b. Integrate $f(x, y, z) = x^2 + y + 3z$ over the line segment joining $(0, 1, 2)$ to $(-1, -1, -2)$.

4. Let C be a curve that encloses a region R such that the area of the region R is 10π and the interior of the region contains the unit disk $D = \{(x, y) : x^2 + y^2 \leq 1\}$. Compute the integral

$$\oint_C \frac{x - 2y}{x^2 + y^2}\, dx + \left(\frac{2x + y}{x^2 + y^2} + 3x\right) dy .$$

(Hint: You might want to use Green's Theorem to compute this integral, but note the problems about the point $(x, y) = (0, 0)$. So you have to use Green's Theorem carefully.)

5. a. Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where C is the intersection of $z = x^2 + y^2 + 1$ and $z = 2y + 1$ oriented clockwise as viewed from above and $\mathbf{F} = < \sin(x^2), y^3, z \ln z - x >$.

b. Let Q be the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane. Find the outward flux of the vector field $\mathbf{F} =< x^3, y^3, z >$ over the boundary of Q.

Spring 2007 Midterm I

1. Determine if each of the following series is convergent or divergent.

a. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \cdot \ln n \cdot (\ln(\ln n))^3}$

b. $\displaystyle\sum_{n=2}^{\infty} \frac{(\ln n)^n}{n}$

c. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \int_n^{n+1} \frac{e^{-x}}{x}\, dx$

2. Determine if each of the following series is convergent or divergent.

a. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^n}{n!}$

b. $\displaystyle\sum_{n=0}^{\infty} \frac{(4n)!}{10^n n!(3n)!}$

c. $\displaystyle\sum_{n=0}^{\infty} \pi^n \sin^2(2^{-n})$

3. Consider the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(n-1)\sqrt{n+1} + (n+1)\sqrt{n-1}}$.

a. Show that this series is convergent.

b. Find the sum of this series by interpreting it as a telescoping series.

4a. Find all solutions of the equation $\displaystyle\frac{x}{4} = \sum_{n=1}^{\infty} \frac{n(n+1)}{x^n}$.

4b. Show that if $1 > a_n > 0$ for all $n \geq 1$ and the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then the series $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{1 - a_n}$ converges.

5a. Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(n-1)(n+1)}{n!}$ exactly.

5b. Determine whether the improper integral $\displaystyle\int_0^1 \frac{dx}{x-\sin x}$ converges or diverges.

Spring 2007 Midterm II

1. Find and classify the critical points of $f(x,y) = x^2y^2 - 2x^2 + x^3 - 4y^2$.

2. Find the equations of **(a)** the tangent plane and **(b)** the normal line to the surface $x^2 + 3y^2 - 4z^2 = 5$ at the point $P(3,2,-2)$.

3. Find the points on the surface $xyz = 1$ closest to the origin.

4a. The derivative of a differentiable function $f(x,y)$ at a point P in the direction of $\mathbf{i}+\mathbf{j}$ is 2, and in the direction of $3\mathbf{i}-4\mathbf{j}$ is $-3/\sqrt{2}$. Find the derivative of f at P in the direction of $7\mathbf{i}-\mathbf{j}$.

4b. Let $h(t)$ be a differentiable function of t and let $g(x,y) = 3h(x^2-y^2) - h(2xy)$. Find $h'(3)$ if $(\nabla g)_{(2,1)} = 6\mathbf{i} + 5\mathbf{j}$.

5. Evaluate the following integrals.

a. $\displaystyle\int_0^{\pi^2} \int_{\sqrt{y}}^{\pi} x^5 \sin\left(\frac{x^2y}{\pi^3}\right) dx\, dy$

b. $\displaystyle\iint_R \sin(x+y)\, dA$ where $R = \{(x,y) : x+y \le \pi/2,\ x \ge 0 \text{ and } y \ge 0\}$

Spring 2007 Final

1. Compute the inward flux of the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + \mathbf{k}$ across the closed surface S composed of the portion of the paraboloid $z = x^2+y^2$ lying below the plane $z = 4$, and the portion of the plane $z = 4$ lying inside the paraboloid $z = x^2 + y^2$.

2a. Suppose $f(x,y)$ is a function with continuous second order partial derivatives on the entire plane, and $f_{xx} + f_{yy} = e^{x^2+y^2}$ for all (x,y). Evaluate the line integral

$$\oint_C \nabla f \cdot \mathbf{n}\, ds$$

where C is the circle $x^2+y^2 = 1$ parametrized counterclockwise, and $\mathbf{n}$ is the outward unit normal field on C.

2b. Evaluate the integral $\iint_R (x-y)^2\cos(x+y)\,dA$, where R is the square with vertices $(\pi/2, 0)$, $(0, \pi/2)$, $(-\pi/2, 0)$ and $(0, -\pi/2)$, using the change of variables $u = x + y$ and $v = x - y$.

3. Evaluate the integral

$$\iiint_D \frac{1}{(x^2+y^2+z^2)^2}\,dV$$

where D is the region lying outside the cylinder $x^2 + y^2 = 1$ and inside the half-cone $z = \sqrt{x^2+y^2}$.

4a. Estimate the value of the integral $\int_0^1 x\sin(x^3)\,dx$ with an error of magnitude less than 10^{-3} using series.

4b. Evaluate the limit

$$\lim_{x\to 0}\frac{e^x - e^{-x} - 2x}{x - \sin x}$$

using series.

5a. Find the exact sum of the series $\displaystyle\sum_{n=0}^{\infty}\frac{3^{2n+1}}{(2n+1)5^{2n+1}}$.

5b. Suppose that $g(t)$ is a differentiable function of t, and $h(x,y) = y\,g(y/x)$. Find $h_y(3,2)$ if $h(3,2) = 7$ and $h_x(3,2) = 4$.

Spring 2006 Midterm I

1. Determine whether each of the following is convergent or divergent.

a. $\displaystyle\int_0^1 \frac{dx}{e^x - e^{-x}}$

b. $\displaystyle\int_1^\infty \frac{dx}{e^x - e^{-x}}$

c. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{1+2+\cdots+n}$

d. $\displaystyle\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n}\right)$

e. $\displaystyle\sum_{n=1}^{\infty}\frac{n^5}{3^n}$

f. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n}{4^n + n^2}$

g. $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{5^n (n!)^2}$

h. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \arctan n$

2. Let $u_n = \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}$ for $n \geq 1$.

a. Show that $\dfrac{1}{2n} \leq u_n$ for all $n \geq 1$.

b. Show that $u_n < \dfrac{1}{\sqrt{n}}$ for all $n \geq 1$.

c. Show that $u_n > u_{n+1}$ for all $n \geq 1$.

d. Show that the series $\displaystyle\sum_{n=1}^{\infty} u_n$ diverges.

e. Show that the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ converges.

f. Choose the correct statement: (*No explanation is required for this part.*)

☐ The series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ converges absolutely.

☐ The series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ converges conditionally.

Spring 2006 Midterm II

1. For each of the following, write the exact sum of the series in the box if it converges and write DIVERGES if it diverges. No further explanation is required. No partial credit will be given.

a. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{3^n}{n!} =$ ☐

b. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{4^n\pi^{2n}}{9^n(2n)!} =$ ☐

c. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{2^n(n+1)} =$ ☐

d. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n(2n+1)} =$ ☐

2. Consider the power series $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n^2+2^n}$.

a. Find the radius of convergence of the series.

b. Determine whether the series converges absolutely, converges conditionally or diverges at the left endpoint of its interval of convergence.

c. Determine whether the series converges absolutely, converges conditionally or diverges at the right endpoint of its interval of convergence.

3. Assume that $f(x, y, z)$ is a differentiable function and $P(2, 1, -1)$ is a point such that

- $(\nabla f)_P$ is parallel to the xz-plane,
- The derivative of f at P in the direction of $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ is 2,
- The derivative of f at P in the direction of $\mathbf{B} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ is -7.

a. Find $(\nabla f)_P$.

b. In which direction does f increase fastest at P? What is the rate change of in this direction?

4. Suppose that $f(x, y)$ is a twice-differentiable function satisfying $f(1, 2) = 3$, $f_x(1, 2) = 5$, $f_y(1, 2) = -4$, $f_{xx}(1, 2) = 7$, $f_{xy}(1, 2) = f_{yx}(1, 2) = -1$ and $f_{yy}(1, 2) = -2$.

a. Find $\displaystyle\frac{d}{dt}f(t^2, 2t^3)\bigg|_{t=1}$.

b. Assume that $g(t)$ is a twice-differentiable function such that $g(1) = 2$ and $f(t, g(t)) = 3$ for all t. Find $g''(1)$.

5. Find the absolute maximum and the absolute minimum values of $f(x, y) = 4x^3 + 9y^2 - 18xy$ on the square $R = \{(x, y) : 0 \le x \le 2 \text{ and } 0 \le y \le 2\}$.

Spring 2006 Final

1. For each of the following, write the exact sum of the series in the box if it converges and write DIVERGES if it diverges. No further explanation is required. No partial credit will be given.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} =$ ☐

b. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n n} =$ ☐

c. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n n!} =$ ☐

d. $\displaystyle\sum_{n=0}^{\infty} (-1)^{n(n+1)/2} \frac{\pi^n}{4^n n!} =$ ☐

2. Find the points P_0 on the paraboloid $z = x^2 + y^2$ such that the tangent plane to the paraboloid at P_0 passes through the points $(1, 0, 0)$ and $(0, 2, 0)$.

3. Evaluate the following iterated integrals.

a. $\displaystyle\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{dy\,dx}{y^4 + 1}$

b. $\displaystyle\int_0^2 \int_{-\sqrt{1-(y-1)^2}}^0 \frac{x^2}{y}\,dx\,dy$

4. Express the triple integral

$$\iiint_D \frac{z}{(x^2 + y^2 + z^2)^{5/2}}\,dV$$

where $D = \{(x, y, z) : x^2+y^2 \le 1 \text{ and } z \ge 1\}$ as an iterated integral in **(a)** cylindrical and **(b)** spherical coordinates, and **(c)** evaluate it using any coordinate system you wish.

5a. Evaluate the line integral

$$\oint_C \frac{-y\,dx + x\,dy}{1 + x^2 + y^2}$$

where C is the unit circle $x^2 + y^2 = 1$.

5b. Find a function $f(x, y)$ such that

$$\oint_C \frac{-y\,dx + x\,dy}{1 + x^2 + y^2} = \iint_R f(x, y)\,dA$$

for every simple closed curve C in the plane and region R it encloses.

Spring 2005 Midterm I

1. Determine whether each of the following series is convergent or divergent. State clearly the name and the conditions of the test you are using.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{2n^4 + 1}}$ **b.** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

c. $\displaystyle\sum_{n=1}^{\infty} \frac{n\,2^n}{3^n}$ **d.** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n\,3^n}$

e. $\displaystyle\sum_{n=0}^{\infty} \frac{(3n)!}{7^n n!(2n)!}$ **f.** $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{\sqrt{n^2 + 1}}$

2. Consider the power series

$$\sum_{n=0}^{\infty} (-1)^n \frac{3^n x^n}{n^2 + 1} \quad .$$

a. Find the radius of convergence of the power series.

b. Determine whether the power series is absolutely convergent, conditionally convergent or divergent at the left endpoint of its interval of convergence.

c. Determine whether the power series is absolutely convergent, conditionally convergent or divergent at the right endpoint of its interval of convergence.

3. Find the sums of the following series exactly:

a. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ **b.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n\, n(n+1)}$

4. In each of the following, if there exists a sequence $\{a_n\}$ which satisfies the given condition, then give an example of such a sequence; otherwise write DOES NOT EXIST. No further explanation is required.

a. $0 < \dfrac{a_{n+1}}{a_n} < 1$ for all $n \geq 1$ and $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

b. The sequence $\{a_n\}$ is convergent and $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

c. $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent and $\displaystyle\sum_{n=1}^{\infty} (a_n)^2$ is convergent.

d. $\dfrac{1}{n} < a_n$ for all $n \geq 1$ and $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

e. $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent and $\displaystyle\sum_{n=1}^{\infty} (a_n)^2$ is divergent.

Spring 2005 Midterm II

1. Let $f(x,y,z) = x^2y - y^3z + xz$, $P_0(1,2,3)$ and $\mathbf{A} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

a. Find the directional derivative of f at P_0 in the direction of $\mathbf{A}$.

b. Find the direction at P_0 in which f increases the fastest.

2. Find and classify the critical points of the function $f(x,y) = y^2 - x^2y - \dfrac{1}{2}\,y^4$.

3. Let $x = s^2t$, $y = st^2$ and $z = f(x,y)$ where f is a function with continuous second order partial derivatives satisfying

$$f(4,2) = 1\,, \qquad f_x(4,2) = -3\,, \qquad f_y(4,2) = 5\,,$$
$$f_{xx}(4,2) = -7\,, \qquad f_{xy}(4,2) = 11\,, \qquad f_{yy}(4,2) = -13\,.$$

Find $\left.\dfrac{\partial^2 z}{\partial s^2}\right|_{(s,t)=(2,1)}$.

4a. Evaluate the iterated integral $\displaystyle\int_0^1 \int_{\sqrt{x}}^1 e^{y^3}\,dy\,dx$.

4b. Evaluate the integral $\displaystyle\iint_R \frac{x}{\sqrt{x^2 - y^2}}\,dA$ where $R = \{(x,y) : 1 \leq x^2 + y^2 \leq 4 \text{ and } -x \leq y \leq x\}$.

5a. Find the volume of the region in the first octant bounded by the cylinders $z = y^2$ and $z = 1 - x^2$, and the coordinate planes.

5b. Evaluate the integral $\displaystyle\iiint_D \frac{1}{(x^2+y^2+z^2)^2}\,dV$ where D is the region inside the cylinder $x^2 + y^2 = 1$ and above the hemisphere $z = \sqrt{2 - x^2 - y^2}$.

Spring 2005 Final

1. Determine whether each of the following series is absolutely convergent, conditionally convergent or divergent.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{101^{102n}}{n!}$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

c. $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}$

d. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

e. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n(n+1)/2}}{n^2}$

2a. Find $(\nabla f)_{(1,2)}$ where $f(x, y) = xy^2 + x^3 - y$.

2b. Find the equation of the tangent plane to the surface $z = xy^2 + x^3 - y$ at $(1, 2, 3)$.

2c. Find the equation of the tangent line to the curve $xy^2 + x^3 - y = 3$ at $(1, 2)$.

3. Let V be the volume of the region bounded on the sides by the cylinder $x^2+y^2 = 3$, at the top by the plane $z = 1$ and at the bottom by the plane $z = 0$. In each of the following, fill in the boxes so that the right side of the equality becomes the iterated integral for V in the corresponding coordinate system and the order of integration.

a. $$V = \int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} \square \; dz\, dy\, dx$$

b. $$V = \int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} \square \; dz\, dr\, d\theta$$

c. $V = \displaystyle\int_{\boxed{}}^{\boxed{}} \int_{\boxed{}}^{\boxed{}} \int_{\boxed{}}^{\boxed{}} \boxed{}\, d\rho\, d\phi\, d\theta$

$\displaystyle + \int_{\boxed{}}^{\boxed{}} \int_{\boxed{}}^{\boxed{}} \int_{\boxed{}}^{\boxed{}} \boxed{}\, d\rho\, d\phi\, d\theta$

4a. Let $u = x^2 - y^2$ and $v = 2xy$. Compute the Jacobian $\dfrac{\partial(u,v)}{\partial(x,y)}$.

4b. Evaluate the integral $\displaystyle\iint_R (x^2+y^2)\, dA$ where $R = \{(x,y) : -1 \le x^2 - y^2 \le 1\,,\ xy \le 1\,,\ x \ge 0 \text{ and } y \ge 0\}$.

5. S : The surface cut from the cone $z^2 = x^2 + y^2$, $z \ge 0$, by the cylinder $x^2 + y^2 = 2x$

$\mathbf{n}$: The unit normal vector field on S pointing away from the positive z-axis

C : The curve of intersection of the cone $z^2 = x^2 + y^2$, $z \ge 0$, and the cylinder $x^2 + y^2 = 2x$, parametrized counterclockwise as seen from a point on the positive z-axis

$\mathbf{F} = yz\,\mathbf{i} - xz\,\mathbf{j}$

a. Find the area of S.

b. Choose and evaluate one of the integrals $\displaystyle\iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\, d\sigma$ and $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r}$.

c. Use the result of part **(b)** to find the value of the other integral.

6. Show that there is a constant c such that every region D in space enclosed by an oriented surface S with outward unit normal vector field $\mathbf{n}$ satisfies the equality

$$\iint_S \mathbf{r}\cdot\mathbf{n}\, d\sigma = c\,V$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and V is the volume of D.

Spring 2004 Midterm I

1. Find all values of the constant p for which the improper integral

$$\int_0^\infty \frac{dx}{x^p\sqrt[3]{x^2+1}}$$

converges.

2. Determine whether each of the following series is convergent or divergent:

a. $\displaystyle\sum_{n=2}^{\infty} \sin\left(\frac{\pi}{n}\right)$

b. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{\pi^n}$

3. Determine whether each of the following series is convergent or divergent:

a. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \ln\left(\frac{n}{n+1}\right)$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[n]{n}}$

4. Determine whether each of the following series is convergent or divergent:

a. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n (n!)^2}{(2n+1)!}$

b. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

5a. Determine the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$ and determine the type of convergence at each point.

5b. How many terms of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ should be used to estimate its sum with an error less then 0.01?

Spring 2004 Midterm II

1a. Find the Taylor series generated by $f(x) = \sin x$ centered at $x = \dfrac{\pi}{4}$.

1b. Show that this series converges to $f(x)$ for all x.

2. Assume that

$$xe^z + y^2 z = \sin x + 1$$

defines x as a differentiable function of y and z and find $\dfrac{\partial x}{\partial y}$ and $\dfrac{\partial x}{\partial z}$ at the point $(x, y, z) = (0, -1, 1)$.

3. Find and classify the critical points of $f(x, y) = x^3 - 3xy + y^3$.

4. Consider the function $f(x, y, z) = x^3 z + \ln(x^2 + y^2) + cz^2$ where c is a constant.

a. Find the value of c if the tangent plane to the level surface $f(x, y, z) = f(1, -1, 2)$ at the point $P_0(1, -1, 2)$ passes through the origin.

b. Let $c = 1$. Find the directional derivative of f at $P_0(1, -1, 2)$ in the direction of $\mathbf{A} = -5\mathbf{i} + \mathbf{j} + 7\mathbf{k}$.

5. Find $\left.\dfrac{\partial^2 w}{\partial t \partial s}\right|_{(t,s)=(2,1)}$ if $x = t^2$, $y = ts$, $z = s^2$, and $w = f(x, y, z)$ is an infinitely differentiable function satisfying:

$$
\begin{array}{lll}
f_x(4,2,1) = 2 & f_y(4,2,1) = -3 & f_z(4,2,1) = 5 \\
f_{xx}(4,2,1) = -7 & f_{xy}(4,2,1) = 11 & f_{xz}(4,2,1) = -13 \\
f_{yy}(4,2,1) = 17 & f_{yz}(4,2,1) = -19 & f_{zz}(4,2,1) = 23
\end{array}
$$

Spring 2004 Final

1. Prove that

$$\int_0^1 \frac{\ln(1+x)}{x}\, dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

and determine how many terms should be used to estimate this sum with an error less than 10^{-2}.

2. Find the absolute maximum and the absolute minimum values of the function $f(x, y) = (y - x^2)(y - 2x^2)$ on the square $R = \{(x, y) : |x| \le 1 \text{ and } |y| \le 1\}$.

3. Find the absolute minimum value of the function $f(x, y, z) = x^2 + y^2 + z^2$ on the surface $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1$ for $x, y, z > 0$.

4. Evaluate the following integrals:

a. $\displaystyle\iint_R xe^{-y}\, dA$ where $R = \{(x, y) : 1 \le x^2 + y^2 \le 4 \text{ and } x \ge 0\}$.

b. $\displaystyle\int_0^{\pi^3/8}\int_{2/\pi}^{1/\sqrt[3]{y}} \sin(x^2 y)\,dx\,dy$

5. Let C be the unit circle. Find the value of the constant a such that

$$\oint_C \big((x+ay)\,dx + (x^2+3x+y)\,dy\big) = 0 \ .$$

where C is oriented counterclockwise.

Spring 2003 Midterm I

1a. Find the equation of the line of intersection of the planes $4x+y+z=0$ and $2x+3y-2z=-5$.

1b. Find the distance between the planes $x-2y+z=3$ and $x-2y+z=4$.

2. Find the point of intersection of the plane passing through the points $P_1(0,-2,-6)$, $P_2(-1,1,5)$ and $P_3(2,3,-6)$, and the line passing through the points $P_4(2,-1,0)$ and $P_5(3,-4,3)$.

3. Determine whether each of the following improper integrals is convergent or divergent.

a. $\displaystyle\int_1^{\infty} \frac{\sqrt[3]{x^4-1}}{x^3}\,dx$ **b.** $\displaystyle\int_0^1 \frac{dx}{\ln x}$

4. Evaluate the following integrals:

a. $\displaystyle\int_0^1 \sqrt{x}\ln x\,dx$ **b.** $\displaystyle\int \frac{dt}{t+\sqrt{1-t^2}}$

5. Evaluate the improper integral

$$\int_0^{\infty} \frac{dx}{(ax^2+1)(x+a)}$$

where a is a positive constant.

Spring 2003 Midterm II

1. Find and classify the critical points of the function $f(x,y)=3x^2y+y^3-108y$.

2. Find the directional derivative of $f(x,y,z)=3x^2yz+2yz^2$ at $P_0(1,1,1)$ in a direction normal to the surface $x^2-y+z^2=1$.

3. Find the absolute maximum value of $f(x,y,z) = xy^3z^5$ on the sphere $x^2+y^2+z^2=1$ using the Lagrange multipliers method.

4a. Determine if the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin^2 x \sin^2 y}{(x^2+y^2)^2}$ exists.

4b. Estimate the change in $z = \ln(x^2+y^2)$ corresponding to the change $dx = 0.2$ and $dy = -0.1$ from $(x,y) = (3,4)$.

5a. Find the values of the constant c for which $w = e^{-2t}\sin cx \cos y$ satisfies $5w_t = w_{xx} + w_{yy}$ for all (x,y,t).

5b. Find f_{xx} at the point $(x,y) = (\sqrt{3},1)$ if $z = f(x,y), x = r\cos\theta, y = r\sin\theta$, and $z_r = 1, z_\theta = -2, z_{rr} = 0, z_{r\theta} = 0, z_{\theta\theta} = 0, z_{rrr} = 3, z_{rr\theta} = -5, z_{r\theta\theta} = 7, z_{\theta\theta\theta} = -11$ at this point.

Spring 2003 Final

1a. Find the points $P_0(x_0,y_0,z_0)$ on the sphere $x^2+y^2+z^2=9$ such that the tangent plane to the sphere at P_0 passes through the points $(4,0,1)$ and $(0,0,9)$.

1b. Find the cosine of the acute angle between the tangent planes to the paraboloid $2z = x^2+y^2$ at the points of intersection of the paraboloid and the line $x = t, y = -t, z = t+2, -\infty < t < \infty$.

2. Find the values of the constant k for which the function $f(r,\theta) = r^k\cos(5\theta)$ satisfies the equation $f_{xx} + f_{yy} = 0$ for all $(x,y) \neq (0,0)$ where $x = r\cos\theta$ and $y = r\sin\theta$.

3. Evaluate the following integrals:

a. $\displaystyle\int_0^{\pi^3}\int_{1/\pi}^{1/\sqrt[3]{y}} \cos(x^2y)\,dx\,dy$

b. $\displaystyle\int_0^{\infty}\int_0^{x} \frac{1}{(1+x^2+y^2)^2}\,dy\,dx$

4. Evaluate $\displaystyle\iint_R y^{-1}(e^x+e^{-x})^{-2}\,dA$ where $R = \{(x,y) : e^x \le y \le 4e^x \text{ and } e^{-x} \le y \le 4e^{-x}\}$.

5. Find the values of the constants a and b for which the limit

$$\lim_{x\to 0} \frac{x^{-3}\sin(ax+x^3)-1}{(1+x^2)\ln(1+bx^2)-2x^2}$$

is a nonzero real number and compute this number.

Spring 2002 Midterm I

1. Determine if each of the following series is convergent or divergent.

a. $\displaystyle\sum_{n=1}^{\infty} \sin(\frac{\pi n}{2})\ \sin(\frac{2}{\pi n})$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n^{100}}{3^n}$

2. Consider the series

$$\sum_{n=4}^{\infty} \frac{1}{n^2-9}$$

a. Find s_n.

b. Find the sum of the series.

3. Find the radius of convergence R and the interval of convergence I of the power series

$$\sum_{n=0}^{\infty} \frac{(x-3)^n}{4^n+n^3}$$

and determine the type of convergence at each point of I.

4. Find the values of the following expressions:

a. $\displaystyle\lim_{x\to 0} \frac{e^{x^2}-1-x^2}{(1+x^4)^{1/2}-1}$

b. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+3}(n+2)}$

5. a. Show that if $\sum_{n=1}^{\infty} a_n$ is convergent and $a_n \geq 0$ for all $n \geq 1$, then $\sum_{n=1}^{\infty} a_n^2$ is also convergent.

b. Give an example of a convergent series $\sum_{n=1}^{\infty} a_n$ for which $\sum_{n=1}^{\infty} a_n^2$ is divergent.

Spring 2002 Midterm II

1. Let $f(x,y) = cye^{xy} + (x+1)^2\cos(\pi y)$ where c is a real constant and $\mathbf{A} = 3\mathbf{i} - 4\mathbf{j}$. Find c if the directional derivative of f at $(0,1)$ in the direction of $\mathbf{A}$ is 2.

2. Find and classify the critical points of $f(x,y) = x^4 + y^4 + 4axy$ where a is a real constant.

3. Find the absolute maximum and minimum values of the function $f(x,y) = xy - x - y + 3$ on the triangular region with vertices at $(0,0), (2,0)$ and $(0,4)$.

4. Find $\left.\dfrac{\partial^2 z}{\partial x \partial y}\right|_{(2,1)}$ if $z = f(u,v)$ where $u = x^2 y$ and $v = x/y$, and f is a twice differentiable function with $f_u(4,2) = 4$, $f_v(4,2) = -5$, $f_{uu}(4,2) = -1$, $f_{uv}(4,2) = f_{vu}(4,2) = 3$, and $f_{vv}(4,2) = 2$.

5. Evaluate the following integrals:

a. $\displaystyle\int_0^1 \int_y^{y^{1/3}} \sin(x^2)\, dx\, dy$

b. $\displaystyle\iint_R \frac{1}{(x^2+y^2)^2}\, dA$ where $R = \{(x,y) : x \geq \sqrt{y^2+1}\,,\ x \geq \sqrt{3}\, y\,,\ y \geq 0\}$.

Spring 2002 Final

1. Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{4^n n^2}$$

Determine the interval of convergence of the series and find the value of $\left.\dfrac{df}{dx}\right|_{x=-1/2}$ explicitly.

2. Let

$$f(x,y,z) = \int_0^{xyz} e^{-t^2}\, dt$$

Compute the value of $f_{xx} + f_{yy} + f_{zz}$ at the point $(2, 1/2, -1)$.

3. Find $\displaystyle\iint_R y^{-3}\, dA$ where $R = \{(x,y) : \sin x \leq y \leq 2\sin x\,,\ \cos x \leq y \leq 2\cos x\,,\ 0 \leq x \leq \pi/2\}$.

4. Let D be the region lying inside the sphere $x^2+y^2+z^2=4$, outside the sphere $x^2+y^2+z^2=z$, and above the xy-plane. Express the triple integral

$$\iiint_D (x^2+y^2+z^2)\,dV$$

in spherical coordinates and evaluate it.

5. Let C be the boundary of $R=\{(x,y) : 0 \le x \le 1\,, 0 \le y \le 1\}$ and $\mathbf{F}=(ax^2+y)\mathbf{i}+xy^2\mathbf{j}$ where a is a constant. Find the value of a for which the outward flux of $\mathbf{F}$ over C and the counterclockwise circulation of $\mathbf{F}$ around C are equal.

Spring 2001 Midterm I

1. Consider the series

$$\sum_{n=1}^{\infty} \ln\left(\frac{n^2+2n+1}{n^2+2n}\right) .$$

a. Show that the series converges.

b. Find the sum of the series.

2. Determine if each of the following series converges or diverges.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{n!\,(2n)!}{(3n)!}$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{e^n-e^{-n}}$

3. Find the sum of each of the following series if it exists. If not, explain why.

a. $\pi-\dfrac{\pi^3}{3!}+\dfrac{\pi^5}{5!}-\dfrac{\pi^7}{7!}+\cdots+(-1)^n\dfrac{\pi^{2n+1}}{(2n+1)!}+\cdots$

b. $\pi-\dfrac{\pi^3}{3}+\dfrac{\pi^5}{5}-\dfrac{\pi^7}{7}+\cdots+(-1)^n\dfrac{\pi^{2n+1}}{2n+1}+\cdots$

4. Find the interval of convergence of the power series

$$\sum_{n=2}^{\infty} \frac{x^n}{n\ln n}$$

and for each point of this interval determine if the convergence is absolute or conditional.

5. Consider the function defined by:

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!\, 2^{n(n-1)/2}}$$

a. Find the domain of f.

b. Evaluate the limit $\lim\limits_{x\to 0} \dfrac{f(x) - e^x}{1 - \cos x}$.

c. Show that $f(2) < e + \dfrac{3}{2}$.

d. Show that $f(-2) < 0$.

Spring 2001 Midterm II

1. Evaluate each of the following limits if it exists, and explain why if it does not.

a. $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{xy+1} - 1}$

b. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2y^2 + (x-y)^2}$

2. Find and classify the critical points of

$$f(x,y) = x^2y + x^2 + y^2 - xy - x \ .$$

3. Consider the function $f(x, yz) = xy^2 + e^{xy} - yz$ and the point $P_0(0, 1, 2)$.

a. Find the direction in which f increases fastest at the point P_0.

b. Find a unit vector $\mathbf{u}$ which is parallel to the xy-plane and which satisfies $\left(\dfrac{df}{ds}\right)_{\mathbf{u},P_0}$.

4. Find the extreme values of $f(x,y,z) = 8x - 4z$ subject to the constraint $x^2 + 10y^2 + z^2 = 5$.

5. Find the volume of the region that lies under the cone $z = (x^2+y^2)^{1/2}$ and above the disk $(x-1)^2 + y^2 \le 1$.

6. a. Change the order of integration in the following iterated integral:

$$\int_1^2 \int_{2-x}^{\sqrt{2x-x^2}} f(x,y)\, dy\, dx \ .$$

b. Express the double integral $\iint_R f(x,y)\,dA$ as an iterated integral in polar coordinates if $R = \{(x,y) : 0 \le y \le x^2\,,\, 0 \le x \le 1\}$.

Spring 2001 Final

1. Determine if each of the following series is convergent or divergent.

a. $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

b. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{2^{n^2}}$

2. Find nonzero real numbers a and b such that the function $f(x,t) = t^a e^{bx^2/t}$ satisfies the equation $\dfrac{\partial f}{\partial t} = \dfrac{\partial^2 f}{\partial x^2}$ for all (x,t) with $t > 0$.

3. Let $D = \{(x,y,z) : x^2 + y^2 + z^2 \le 4 \text{ and } z \ge 1\}$. Express -do not evaluate- the volume of D as an iterated integral in

a. cylindrical coordinates,

b. spherical coordinates.

4. Let $D = \{(x,y,z) : 1 \le x \le 2\,,\, 0 \le xy \le 2\,,\, 0 \le z \le 1\}$. Evaluate the triple integral $\iiint_D (x^2y + 3xyz)\,dV$ by applying the transformation $u = x, v = xy, w = 3z$.

5. Find the area of the piece of the cylinder $x^2+z^2 = 1$ which lies inside the cylinder $x^2 + y^2 = 1$ and in the first octant.

6a. Use Stokes's Theorem to evaluate the line integral $\oint_C y\,dx + z\,dy + x\,dz$ where C is the intersection curve of the plane $x+y+z = 0$ with the sphere $x^2+y^2+z^2 = 4$ parametrized in the counterclockwise direction as seen from the positive z-axis.

6b. Use the Divergence Theorem to find the outward flux of the vector field $\mathbf{F} = xz\mathbf{i}+yz\mathbf{j}+(3z-z^2)\mathbf{j}$ across the boundary of the ball $D = \{(x,y,z) : x^2+y^2+z^2 \le 4\}$.

Spring 2000 Midterm I

1. Find the sum of the series
$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} \quad .$$

2. Find the radius of convergence and the interval of convergence of the power series
$$\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2+3}} \; .$$
Also determine the type of convergence (absolute or conditional) for each x in the interval of convergence.

3. Evaluate the following limit:
$$\lim_{y \to 0} \frac{\arctan y - \sin y}{y^3 \cos y}$$

4. Determine if each of the following limits exist.

a. $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^4 - y^2}{x^4 + y^2}$

b. $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^3 - xy^2}{x^2 + y^2}$

5. Find the values of the constant α such that the function
$$w = (x^2 + y^2 + z^2)^{\alpha}$$
satisfies the equation $w_{xx} + w_{yy} + w_{zz} = 0$ for all $(x, y) \neq (0, 0)$.

Spring 2000 Midterm II

1. Find the directional derivative of the function $f(x, y, z) = \cos(xy) + e^{yz} + \ln(xz)$ at the point $P_0(1, 0, 1/2)$ in the direction of $\mathbf{A} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

2. Find the absolute maximum and minimum values of the function
$$f(x, y) = 6xy - x^3 - 3y^2$$
on the region $R = \{(x, y) : 0 \le x \le 2 \text{ and } 0 \le y \le 1\}$.

3. Evaluate the following integrals:

a. $\displaystyle \int_0^1 \int_{y^{1/3}}^1 \frac{\sin(\pi x^2)}{x^2} \, dx \, dy$

b. $\displaystyle\iint_R \frac{1}{(1+x^2+y^2)^2}\,dA$ where $R = \{(x,y) : x^2+y^2 \le 1\}$.

4. Let D be the region bounded below by the xy–plane, on the sides by the sphere $\rho = 2$ and above by the cone $\phi = \pi/3$. Express (Do not evaluate!) the volume of D in terms of iterated integrals in:

a. spherical coordinates

b. cylindrical coordinates

c. Cartesian coordinates

5. Evaluate the integral

$$\iint_R \frac{1}{1+x^2y^2}\,dA$$

where $R = \{(x,y) : x/2 \le y \le 2x \ \text{ and } \ xy \ge 1\}$.

Spring 2000 Final

1. Determine whether each of the following series is convergent or divergent:

a. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\,\frac{\sin n}{n^2}$

b. $\displaystyle\sum_{n=1}^{\infty}\frac{3^n}{n^2 2^n}$

2. Find the absolute maximum value of the function $f(x,y,z) = x - y + 2z$ on the ellipsoid $x^2+y^2+16z^2 = 16$.

3. Let S be the part of the cone $z^2 = 4x^2+4y^2$ which lies above the $xy-$plane and inside the cylinder $x^2+y^2 = 2x$. Evaluate the surface integral $\displaystyle\iint_S (x^2+y^2)\,d\sigma$.

4. Let D be the region lying in the first octant between the cones $z^2 = 3x^2+3y^2$ and $z^2 = x^2+y^2$, and inside the sphere $x^2+y^2+z^2 = 4$. Let S be the boundary of D . Let $\mathbf{n}$ be the outward pointing unit normal vector field on S .

a. Find the volume of D .

b. Evaluate the integral $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$ where

$$\mathbf{F} = (x^2+\sin y)\mathbf{i} + (e^z - 3xy)\mathbf{j} + (xz+7z)\mathbf{k} \ .$$

5. Let S be the surface $4x^2 + 9y^2 + 36z^2 = 36,\ z \geq 0$. Let $\mathbf{n}$ be the unit normal vector field on S pointing away from the origin. Find $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$ where

$$\mathbf{F} = y\mathbf{i} + x^2\mathbf{j} + (x^2 + y^4)^{3/2} \sin(e^{xyz})\mathbf{k} \ .$$

Spring 1999 Midterm I

1. Find the constants a and b such that the limit

$$\lim_{x \to 0} \frac{\tan^{-1}(x + ax^3 + bx^5) - x}{x^7}$$

exists and is finite. Find the limit in this case. (Do <u>not</u> use L'Hôpital's rule.)

2. Evaluate the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}}{2^n\, n(n+1)} \ .$$

3. Determine the values of the constant α for which the function

$$f(x,y) = \begin{cases} \dfrac{x^4 + y^4}{(x^2 + y^2)^\alpha} & \text{if } (x,y) \neq (0,0), \\ & \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$$

is continuous at $(x,y) = (0,0)$.

4. Assume that $z = f(x,y)$ is a differentiable function of x and y, and $x = g(r,s)$ and $y = h(r,s)$ are differentiable functions of r and s. Use the following table

$f(2,3) = 10$	$f_x(2,3) = 13$	$f_y(2,3) = 16$
$f(2,4) = 11$	$f_x(2,4) = 14$	$f_y(2,4) = 17$
$f(3,5) = 12$	$f_x(3,5) = 15$	$f_y(3,5) = 18$
$g(0,1) = 2$	$g_r(0,1) = 6$	$g_s(0,1) = 0$
$g(1,0) = 3$	$g_r(1,0) = 7$	$g_s(1,0) = 1$
$h(0,1) = 4$	$h_r(0,1) = 8$	$h_s(0,1) = 0$
$h(1,0) = 5$	$h_r(1,0) = 9$	$h_s(1,0) = 1$

to compute

$$\left.\frac{\partial z}{\partial r}\right|_{(r,s)=(0,1)} \quad \text{and} \quad \left.\frac{\partial z}{\partial s}\right|_{(r,s)=(1,0)} \ .$$

5. Find $\partial x/\partial w$ when $(x, y, z, w) = (2, -2, 1, -1)$ if x and z are defined as functions of y and w by the equations:

$$\begin{cases} x^3 + y^2 z + xz^2 w - z^4 = 9 \\ x^2 z^2 - x^5 w + w^3 + y^3 = 27 \end{cases}$$

Spring 1999 Midterm II

1. Find and classify the critical points of the function:

$$f(x, y) = x^3 y + x^3 + 4y^2$$

2. Find the absolute maximum and the absolute minimum values of the function

$$f(x, y) = x^3 + y^3 + 6xy + 8$$

on the rectangular region R bounded by the lines $y = x + 2$, $y = x - 2$, $x + y = 1$ and $x + y = -2$.

3. Evaluate the integral $\displaystyle\int_0^1 \int_0^x \sqrt[3]{(2y - y^2)^2}\, dy\, dx$.

4. Find the volume of the region D which lies between the spheres $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 + z^2 = 2z$, and inside the cone $z = \sqrt{(x^2 + y^2)/3}$.

5. Evaluate the integral $\displaystyle\iint_R \frac{1}{x} e^{\sqrt[3]{xy^2}}\, dA$ where R is the region in the first quadrant bounded by the curves $y = 8x$, $y = 27x$, $y = \dfrac{1}{8\sqrt{x}}$ and $y = \dfrac{1}{\sqrt{x}}$.

Spring 1999 Final

1. Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} n x^{3n+1}$$

and find $f(x)$ explicitly.

2. Evaluate the integral

$$\iiint_D \frac{z}{(x^2 + y^2 + z^2)^{5/2}}\, dV$$

where $D = \{(x, y, z) : x^2 + y^2 \leq z^2 \text{ and } z \geq 1\}$.

3. Find the area of the portion of the cylinder $x^2 + y^2 = 2x$ that lies inside the sphere $x^2 + y^2 + z^2 = 4$ and in the first octant.

4. Verify the Stokes's Theorem for the vector field $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ and the surface $S = \{(x, y, z) : z = 1 - x^2 - y^2 \text{ and } z \geq 0\}$ with the unit normal vector field $\mathbf{n}$ satisfying $\mathbf{n} \cdot \mathbf{k} > 0$.

5. Let $f(x, y, z)$ be a function with continuous second order partial derivatives and assume that $f(x, y, z) \neq 0$ for all (x, y, z). Assume also that $|\nabla f|^2 = 4f$ and $\nabla \cdot (f \nabla f) = 10f$. Evaluate

$$\iint_S \nabla f \cdot \mathbf{n} \, d\sigma$$

where S is the sphere $x^2 + y^2 + z^2 = 1$ and $\mathbf{n}$ is the outward unit normal to S.

Spring 1998 Midterm I

1. Determine whether each of the following series converges or diverges:

a. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n \ln(\ln n)}$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{2n-1} \ln n}{n(n+1)}$

c. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

d. $\displaystyle\sum_{n=0}^{\infty} \frac{\sin((2n+1)\pi/2)}{\sqrt{n+1}}$

2. Find all values of x for which the following series is convergent and also determine the type of convergence:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left(\frac{x+3}{x} \right)^n$$

3. Find the first three non-zero terms of the Maclaurin series of $f(x) = \sin(\sin x)$.

4. a. Find $\displaystyle\int_0^1 \cos x^3 \, dx$ approximately with an error of magnitude less than 5×10^{-4}.

b. Find the sum of the following series as a function of x for $|x| < 1$:

$$\frac{x^2}{2} - \frac{x^3}{3 \cdot 2} + \frac{x^4}{4 \cdot 3} - \frac{x^5}{5 \cdot 4} + \cdots$$

5. a. Find the intersection points of the curves $r = 1 + \cos\theta$ and $r = \cos\theta$.

b. Find the area of the region which lies inside the curve $r = 1+\cos\theta$ and outside the curve $r = \cos\theta$.

Spring 1998 Midterm II

1. Find the equation of the line L which passes through the point $P_0(1, 0, 2)$ and intersects the line $L_1 : \dfrac{x-1}{2} = \dfrac{y+1}{3}, z = 1$ orthogonally.

2a. Let $\vec{r}(t)$ be the position vector of a curve in space and let $\vec{v}(t) = \dfrac{d\vec{r}}{dt}$ be the corresponding velocity vector. Prove that $\vec{v}(t) \cdot \vec{a}(t) = \dfrac{1}{2}\dfrac{d(v(t)^2)}{dt}$, where $\vec{a}(t) = \dfrac{d\vec{v}}{dt}$ is the acceleration vector of $\vec{r}(t)$ and v is the speed (= magnitude of $\vec{v}$).

2b. Let $\vec{A}$ and $\vec{B}$ be two fixed vectors making an angle of $\dfrac{\pi}{3}$ radians with each other and $|\vec{A}| = 2$, $|\vec{B}| = 3$. A particle moves on a space curve C in such a way that its position vector $\vec{r}(t)$ and velocity vector $\vec{v}(t)$ are related by the equation $\vec{v}(t) = \vec{A} \times \vec{r}(t)$ for all $t \in \mathbf{R}$. Moreover assume that $\vec{r}(0) = \vec{B}$. Show that the speed of the particle is constant and find its value.

2c. Prove that the curvature κ of the curve C in part **(b)** is constant and calculate its value.

Hint: $\kappa = \dfrac{|\vec{v}(t) \times \vec{a}(t)|}{|\vec{v}(t)|^3}$.

3a. Find the set of all points on the surface $(y + z)^2 + (z - x)^2 = 16$ where the normal line is parallel to the yz-plane. Describe this set.

3b. Find the extremum points of the function $f(x, y) = x^3 + 3xy + y^3$.

4. Let $s(x)$ be an even differentiable function of $x \in \mathbf{R}$ and

$$f(x) = \begin{cases} \dfrac{s(x) - s(y)}{x^2 - y^2} & , \ y \neq \pm x, \\ \dfrac{\sin x}{x} & , \ y = \pm x \text{ and } x \neq 0, \\ a & , \ (x, y) = (0, 0) \ . \end{cases}$$

be a differentiable function of $(x, y) \in \mathbf{R}^2$.

a. Find $s(x)$ and the constant a.

b. Find $\dfrac{\partial f(x,y)}{\partial x}$ at the origin.

5. Find the points on the curve $5x^2+6xy+5y^2 = 9$ which are nearest to and farthest from the origin.

Spring 1998 Final

1a. Find the set of all real numbers x for which the following series is convergent:

$$\sum_{n=0}^{\infty} \frac{2^{nx}}{n^2+1}$$

1b. Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{5^n(n+1)}$ as a function of x.

2. Evaluate the double integral

$$\iint_R \frac{x^2}{y^4}\, dx\, dy$$

where R is the region in the plane bounded by the curves $xy = 2$, $xy = 4$, $y^2 = x$ and $y^2 = 3x$.

3. Let $\mathbf{F} = (2xy^2z + xy^3)\mathbf{i} + (2x^2yz + \frac{3}{2}x^2y^2)\mathbf{j} + (x^2y^2 + 3z^2)\mathbf{k}$. Evaluate

$$\int_{(1,1,-1)}^{(-1,-1,1)} \mathbf{F} \cdot d\mathbf{r}\ .$$

4. Let $f(x, y)$ and $g(x, y)$ have continuous first order partial derivatives. Let

$$\mathbf{F} = g(x,y)\mathbf{i} + f(x,y)\mathbf{j} \quad \text{and} \quad \mathbf{G} = \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{i} + \left(\frac{\partial g}{\partial x} - \frac{\partial g}{\partial y}\right)\mathbf{j}\ .$$

It is known that for the points (x, y) on the circle $x^2 + y^2 = 1$, we have $f(x, y) = 1$, $g(x, y) = y$. Let R be the region in the plane defined by $x^2 + y^2 \leq 1$. Find

$$\iint_R \mathbf{F} \cdot \mathbf{G}\, dx\, dy\ .$$

5. Verify the Divergence Theorem for the vector field $\mathbf{F} = (z^2 + 2)\mathbf{k}$ and the surface S which consists of the upper half of the sphere $x^2 + y^2 + z^2 = a^2$ together with its base, the disk of radius a centered at the origin in the xy-plane.

Spring 1997 Midterm I

1. Find all values of x for which $\displaystyle\sum_{n=1}^{\infty} \frac{2^n x^n}{n(3x+1)^n}$ converges and determine whether the convergence is absolute or conditional.

2. a. Use the Taylor series of $\tan^{-1} x$ at $x = 0$ to find the first four nonzero terms of the Taylor series of $\tan^{-1}(ax + bx^3)$ at $x = 0$ where a and b are nonzero constants.

b. Find a and b for which the limit

$$\lim_{x \to 0} \frac{\tan^{-1}(ax + bx^3) - x}{x^5}$$

is finite and find the value of this limit. (Do <u>not</u> use L'Hôpital's Rule.)

3. Let $\mathbf{A}$ and $\mathbf{B}$ be vectors in space. Show that if $|\mathbf{a} + x\mathbf{B}| \geq |\mathbf{A}|$ for all real numbers x, then $\mathbf{A}$ and $\mathbf{B}$ are orthogonal.

4. Consider the limit

$$\lim_{(x,y) \to (0,0)} \frac{x|y|^k}{x^2 + y^4}$$

where k is a positive constant.

a. Find all values of k for which the limit does <u>not</u> exist.

b. Find all values of k for which the limit exists. Show the existence of the limit by the ε-δ method.

5. Find the linear approximation to $f(x, y) = xy^2 + x^3 y$ at $(1, 2)$ and find an upper bound for the magnitude of the error over the rectangle $R = \{(x, y) : |x - 1| \leq 1/2, |y - 2| \leq 1\}$.

Spring 1997 Midterm II

1. Find and classify the critical points of the function

$$f(x, y) = x^3 - 3xy + y^3 \, .$$

2. a. Let $h(x,y) = f(x+y) + g(x-y)$ where f and g are twice differentiable one-variable functions. Show that

$$\frac{\partial^2 h}{\partial x^2} = \frac{\partial^2 h}{\partial y^2}$$

for all (x,y).

b. Find $u(x,y)$ if $u(x,0) = 0$ and $\dfrac{\partial u}{\partial x}(x,0) = \dfrac{x}{1+x^2}$ for all x, and $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial y^2}$ for all (x,y).

3. Evaluate the following integrals:

a. $\displaystyle\iint_R \ln(x^2+y^2)\, dA$ where $R = \{(x,y) : x^2+y^2 \le 2 \text{ and } 0 \le y \le \sqrt{3}x\}$.

b. $\displaystyle\int_0^\infty \int_y^\infty e^{-x^2}\, dx\, dy$

4. Evaluate $\displaystyle\int_{-1}^0 \int_{-y}^{2y+3} \frac{x+y}{(x-2y)^2} e^{x-2y}\, dx\, dy.$

5. Sketch the region which lies above the surface $z = 2\sqrt{x^2+y^2}$ and inside the surface $x^2+y^2+z^2 = 4z$, and express its volume as a triple integral in **(a)** Cartesian, **(b)** cylindrical, **(c)** spherical coordinates. (Do not evaluate.)

Spring 1997 Final

1. Find the first five nonzero terms of the Taylor series of $f(x) = e^{\sin x}$ centered at $x = 0$.

2. a. Find the absolute maximum of $f(x,y,z) = \ln x + \ln y + 3\ln z$ on the portion of the sphere $x^2+y^2+z^2 = 5r^2$ where $x > 0$, $y > 0$ and $z > 0$. (Here r is a positive constant.)

b. Use part **(a)** and show that

$$abc^3 \le 27\left(\frac{a+b+c}{5}\right)^5$$

for all positive real numbers a, b and c.

3. Let a be a positive constant. Find the area of the portion of the sphere $x^2+y^2+z^2 = a^2$ lying inside the cylinder $x^2+y^2 = ay$.

4. Evaluate the line integral

$$\oint_C \left(\frac{y^3}{x} - e^x\right) dx + \left(\frac{y}{x} + e^{y^2}\right) dy$$

where C is the boundary of the region lying between the curves $y = x^2$, $y = 2x^2$, $y = 1/x$, $y = 3/x$, traced counterclockwise.

5. a. If S is an oriented closed surface with unit normal field $\mathbf{n}$ and $\mathbf{F}$ is a vector field with continuous derivatives, then show that $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = 0$.

b. Find a vector field whose curl is $2x\mathbf{i} + 3y\mathbf{j} + 5z\mathbf{k}$ or show that there is no such vector field.

Spring 1996 Midterm I

1. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} .$$

a. Find the nth partial sum s_n explicitly.

b. Use part **(a)** to evaluate the sum of the series.

2. Let $\displaystyle\sum_{k=1}^{n} \frac{1}{k} - \ln n$ for $n \geq 1$.

a. Show that $a_n > 0$ for $n \geq 1$.

b. Show that $\{a_n\}$ is a decreasing sequence.

c. Show that the sequence $\{a_n\}$ converges.

3. Consider the power series $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} x^n$.

a. Find the radius of convergence of the series.

b. Show that $(1 - 4x)f'(x) = 2f(x)$ for all x in the interval of convergence.

c. Use part **(b)** to express $f(x)$ explicitly.

4. Consider the identity

$$\frac{1}{1 - t^2} = 1 + t^2 + t^4 + \cdots + t^{2n} + r(t, n)$$

where n is a positive integer.

a. Find $r(t,n)$.

b. Integrate the identity above to obtain $\tanh^{-1} x = \sum_{k=0}^{2n+1} a_k x^k + R(x,n)$ for $|x| < 1$ and find a_k for $0 \le k \le 2n+1$.

c. Show that $\lim_{n\to\infty} R(x,n) = 0$ when $|x| < 1$.

5. Determine whether each of the following series or convergent or divergent:

a. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{\ln n}}$

c. $\displaystyle\sum_{n=1}^{\infty} \left(1 - \frac{1}{n^2}\right)^n$

d. $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n}$

Spring 1996 Midterm II

1. Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

2. Find the parametric equations for the line that is tangent to the curve of intersection of the surfaces $x^3 + 3x^2y^2 + y^3 + 4xy - z^2 = 0$ and $x^2 + y^2 + z^2 = 11$ at the point $(1,1,3)$.

3. Find all maxima, minima and saddle points of the function $f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$ in the entire plane.

4. Find the absolute minimum and maximum values of the function $f(x,y) = 4x - 8xy + 2y + 1$ on the triangular region bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$ in the first quadrant.

5. Find the points on the sphere $x^2 + y^2 + z^2 = 25$ where $f(x,y,z) = x + 2y + 3z$ has its maximum and minimum values.

Spring 1996 Final

1. Consider the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} .$$

a. Find the radius of convergence of the series.

b. Evaluate the sum of the series at both end points of the interval of convergence.

2. Evaluate the following integrals:

a. $\displaystyle\int_0^1 \int_y^1 x^2 e^{xy}\, dx\, dy$

b. $\displaystyle\iint_D \frac{1}{x^2+y^2+z^2+1}\, dV$ where D is the intersection of the unit ball with the first octant.

3. Let $R = \{(x,y) : x \le y \le 2x, 1/x \le y \le \sqrt{3}/x\}$. Evaluate the integral $\displaystyle\int_R \frac{1}{1+x^2y^2}\, dA$ by using a change of variables which transforms R into a rectangle.

4. Let D be the region bounded by the surfaces $z = x^2+y^2+1$ and $z^2 = 4x^2+4y^2$. Find the volume of D.

5. Let C be the curve which traces the graph of $y^2 = (1-x^2)(1+x^2)(1+x^4)(1+x^6)$ once counterclockwise. Let $\mathbf{F}(x,y) = \dfrac{x-y}{x^2+y^2}\mathbf{i} + \dfrac{x+y}{x^2+y^2}\mathbf{j}$. Evaluate the integral $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r}$.

6. Suppose $f(x,y)$ is a function on $R = \{(x,y) : (x,y) \ne (0,0)\}$ which has continuous second order partial derivatives and satisfies the equations

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial f}{\partial r} = 0$$

throughout R, where $x = r\cos\theta$ and $y = r\sin\theta$.

a. Find $\dfrac{\partial^2 f}{\partial \theta^2}$.

b. Find $f(0,1)$ if $f(1,0) = 5$.

Spring 1995 Midterm I

1a. Explain whether $\displaystyle\sum_{n=1}^{\infty} (\sin\frac{1}{2n} - \sin\frac{1}{2n+1})$ converges or diverges.

1b. Find the interval of convergence of $\sum_{n=1}^{\infty} n\tan(1/n)x^n$.

2a. Given $\mathbf{A} = \mathbf{i}+\mathbf{j}-2\mathbf{k}$, $\mathbf{B} = -\mathbf{i}-\mathbf{k}$, $\mathbf{C} = 2\mathbf{i}+4\mathbf{j}-2\mathbf{k}$, find the volume of the parallelepiped determined by **A**, **B** and **C**.

2b. Find the area of the projection of the parallelogram determined by **B** and **C** into the xz-plane.

2c. Find the planes determined by **B** and **C**.

3a. Find the area of the region between the circles $r = 1$ and $r = 2\cos\theta$.

3b. Plot the graph of $r = e^{\theta}$ for $0 \le \theta \le 2\pi$, and calculate its length.

4. Evaluate $\displaystyle\int_0^{0.1} \frac{\sin x}{x}\,dx$ within an error of magnitude less than 10^{-11}.

5. Evaluate

$$\lim_{x\to 0} \frac{1-x+x^4+\ln(1+x-x^2)-\cos(\sqrt{3}x)}{x^{5/2}\sin\sqrt{x}} .$$

Spring 1995 Final

1. Let $\mathbf{F} = y^2/x\mathbf{i} + 2y\ln x\mathbf{j} - 2/z^2\mathbf{k}$.

a. Show that **F** is a conservative vector field.

b. Find the potential function.

c. Evaluate $\displaystyle\int_{(1,1,1)}^{(2,2,2)} \mathbf{F}\cdot d\mathbf{r}$.

d. Under what circumstances is your answer to part **(c)** valid?

2a. Evaluate the surface integral $\displaystyle\iint_S (x+z)\,d\sigma$ where S is the first octant portion of the cylinder $z^2+y^2 = 9$ between the planes $x = 0$ and $x = 1$.

2b. Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive numbers. Does the series $\sum_{n=1}^{\infty} \ln(1+a_n)$ converge? Explain your answer.

3. Evaluate $\iint_R e^{(x-y)/(x+y)}\,dx\,dy$, where R is the region bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$.

Hint: Use the transformation $u = x - y$, $v = x + y$.

4a. Find the extremal values of $f(x, y, z) = x - 2y + 2z$ among the points (x, y, z) with $x^2 + y^2 + z^2 = 9$.

4b. Find the area of the triangle with vertices located at the points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$.

5. Let $\mathbf{F} = (x\mathbf{i} + y\mathbf{j})/r$ where $r^2 = x^2 + y^2$. Calculate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ when

a. C is enclosing the origin.

b. C is not enclosing the origin.

Spring 1994 Midterm I

1. Given the series

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1} - \frac{n+2}{n+3} \right)$$

a. Find a formula for the partial sums s_n.

b. Find the sum of the series.

2. Determine whether absolutely convergent, conditionally convergent or divergent. Explain.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{n^2+1}}$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n \ln n \ln(\ln n)}$

3. find the domain of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{4^n n}.$$

4. a. [*missing*]

b. Approximate the value of the following integral with error less than 0.0001:

$$\int_0^{0.3} \cos x^2 \, dx$$

Hint: Use alternating series estimate.

5. Consider the curves $r = -1 + \cos\theta$ and $r = 3\cos\theta$.

a. Sketch the curves.

b. Find the intersection points of these curves in polar and Cartesian coordinates.

c. Find the length of the curve $r = 3\cos\theta$ between two intersection points. (Choose any two you want.)

6. a. Find the length of the curve $x = 2e^t$, $y = e^{3t}/3 + e^{-t}$, $0 \le t \le 1$.

b. Find the area of the region inside $r = 1 + \cos\theta$ and outside $r = 1$.

Spring 1994 Midterm II

1. Find the shortest distance between the lines L_1 and L_2 where

$$L_1 : \begin{cases} x = 1 + 2t \\ y = -1 + t \\ z = t \end{cases} \quad \text{and} \quad L_2 : \begin{cases} x = 2 + s \\ y = 1 - s \\ z = 2s \end{cases} .$$

2a. Find $\lim_{(x,y)\to(0,0)} \arctan\left(\dfrac{x-y}{x^2+y^2}\right)$ or show that it does not exist.

2b. Suppose $w = x^3 - x^2y^5 + 3z + 2t$ and $x + 5z + 3t = 10$ find all possible values of $\dfrac{\partial w}{\partial x}$.

3a. Find the angle between ∇u and ∇v at all points with $x \neq 0$ and $y \neq 0$ if

$$x = e^u \cos v \quad \text{and} \quad y = e^u \sin v \, .$$

3b. Show that the curve $\mathbf{r} = \sqrt{t}\mathbf{i} + \sqrt{t}\mathbf{j} - (t+3)/4\mathbf{k}$ is normal to the surface $x^2 + y^2 - z = 3$ at their intersection point.

4. A particle moves with position vector

$$\mathbf{r} = t\mathbf{A} + t^2\mathbf{B} + \frac{4}{3}t^{3.2}\mathbf{a} \times \mathbf{B} \, , \; t \ge 0$$

where **A** and **B** are two fixed unit vectors making an angle of $\frac{\pi}{2}$ radians with each other.

a. find the speed of the particle at time t.

b. How long does it take for the particle to move 12 units of arc length from the initial position $\mathbf{r}(0)$?

c. Find the curvature at time $t > 0$.

5. Let $f(x, y) = cos(y - e^x)$.

a. Find the linear approximation of $f(x, y)$ at the point$(0, 1)$ and estimate the error made in this approximation if $|x| \leq 0.1$ and $|y - 1| \leq 0.1$.

b. Find the quadratic approximation of the same function at $(0, 1)$.

6. Find the absolute extreme values of $f(x, y) = x^4 + y^4 - 4xy$ in the region bounded by the lines $x = 2$, $y = -2$ and $y = x$.

Spring 1994 Final

1. The plane $x + y + 2z = 0$ intersects the sphere $x^2 + y^2 + z^2 = 1$ along a curve C. Find

$$\oint_C (x + 2z)\, dx + (x + y + z)\, dy + (x + y + z)\, dz$$

if C is traversed in a direction that is counterclockwise when viewed from high above the xy-plane.

2a. Find the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^3}{(3n)!} x^n$.

2b. Find the sum of the infinite series

$$x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots \quad \text{for } |x| < 1.$$

2c. Is the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ convergent or divergent?

3. Consider

$$\oint_C \left(-\frac{y^3}{(x^2 + y^2)^2}\, dx + \frac{xy^2}{(x^2 + y^2)^2}\, dy \right) .$$

a. Evaluate the above line integral when C is the circle $x^2 + y^2 = a^2$ traversed in the counterclockwise direction.

b. Let C be an arbitrary smooth simple closed curve in the plane that does not pass through the origin. Show that above line integral has two possible values depending on whether the origin lies inside or outside C.

4. Find the volume of the parallelepiped bounded by the six planes $x+y+2z=\pm 3$, $x-2y+z=\pm 2$ and $4x+y+z=\pm 6$.

5. The plane $x+y+z=12$ intersects the paraboloid $z=x^2+y^2$ in an ellipse. Find the highest and lowest points on this ellipse (the points with greatest and least z-coordinates).

6a. Let f be a function of x and y. Express $(f_x)^2+(f_y)^2$ in polar coordinates (in terms of the partial derivatives with respect to the polar variables).

6b. Show that if $w=f(u,v)$ satisfies the Laplace equation $f_{uu}+f_{vv}=0$, and if $u=(x^2-y^2)/2$ and $v=xy$, then w satisfies the Laplace equation $w_{uu}+w_{vv}=0$.

Spring 1993 Midterm I

1. Each of the following series is the value of the Maclaurin series of a function at a point. What function and what point? What is the sum of the series?

a. $1-\dfrac{\pi^2}{9\cdot 2!}+\dfrac{\pi^4}{81\cdot 4!}-\cdots+(-1)^n\dfrac{\pi^{2n}}{3^{2n}(2n)!}+\cdots$

b. $\dfrac{2}{3}-\dfrac{4}{18}+\dfrac{8}{81}-\cdots+(-1)^{n-1}\dfrac{2^n}{3^n n}+\cdots$

2. Use series to find the values of a and b for which the limit

$$\lim_{x\to 0}\frac{x-\sin(ax+bx^3)}{x^5}$$

exists and compute that limit.

3. a. Find the interval of convergence of the series

$$y=1+\frac{1}{6}x^3+\cdots+\frac{1\cdot 4\cdot 7\cdot\cdots\cdot(3n-2)}{(3n)!}x^{3n}+\cdots .$$

b. Show that the function defined by the series satisfies a differential equation of the form

$$\frac{d^2y}{dx^2}=x^a y+b$$

and find the values of the constants a and b.

4. Determine for each of the following series whether it converges or diverges. Give reasons for your answers.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^3+2}}$

b. $\displaystyle\sum_{n=0}^{\infty} n^2 e^{-n} \sin n$

5. a. Using vectors in the plane, find the angle between the tangent to the curve $y = f_1(x)$ at $x = x_1$ and the tangent to the curve $y = f_2(x)$ at $x = x_2$.

b. Write $\mathbf{B} = -\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ as the sum of a vector parallel to $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and a vector orthogonal to it.

6. a. Find the area of the region which lies inside the curve $r = 2b\sin\theta$ and outside the curve $r = a$ where $2b > a$.

b. Find the length of the piece of the curve $r = 2b\sin\theta$ lying outside the curve $r = a$.

Spring 1993 Midterm II

1. a. Find the equation of the line normal to the surface $z = x^2 + 3y^2$ at the point $(1, -1, 4)$. Find the coordinates of all intersection points of this line with the surface.

b. Find the parametric equation for the tangent line to the curve of intersection of the surfaces $z = x^2 + 3y^2$ and $z = 6 - x^2 - y^2$ at the point $(1, -1, 4)$.

2. a. Show that the curvature is given by $\kappa = \dfrac{-1}{\mathbf{r}\cdot\mathbf{N}}$ for a curve on the sphere $x^2 + y^2 + z^2 = R^2$.

b. Show that a curve in space with zero curvature at all points is a straight line.

3. Let

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

a. Is f continuous at $(0,0)$? Prove your statement.

b. Show that $-\dfrac{1}{2} \leq f(x,y) \leq \dfrac{1}{2}$ for all (x,y).

4. Use Lagrange multipliers to find the absolute maximum of $f(x,y,z) = x^3 + 12yz$ on the sphere $x^2 + y^2 + z^2 = 25$.

5. Let $f(x,y) = x^3 + y^2 - 2x$.

a. Find and classify the critical points of f.

b. Find the absolute maximum and minimum of f on the line segment $\{(x,y) : x = t, y = t+1, -1 \leq t \leq 1\}$.

6. Let $u(x,y)$ and $v(x,y)$ be two functions with continuous second partial derivatives which satisfy the differential equations $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.

a. Show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ and $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$.

b. Show that all critical points of u are saddle points. You may assume that at a critical point at least one of the second partial derivatives is not zero.

Spring 1993 Final

1. a. Find the absolute minimum of the function defined by the series $\sum_{n=0}^{\infty} n^2 x^n$ on the interval $(-1, 0]$.

b. Find the radius of convergence of the series $\sum_{n=1}^{\infty} \dfrac{n!x^n}{n^n}$.

2. Let r and θ be the polar coordinates in the plane. Suppose $f(x,y)$ has continuous second partial derivatives. Express $f_{xx} + f_{yy}$ in terms of r and θ, and partial derivatives of f with respect to r and θ.

3. Let R be the region bounded by the curves $xy = 1$, $xy = 4$, $y = \sqrt{3}x^2$ and $y = x^2$ in the first quadrant. Evaluate

$$\iint_R \frac{x^2 y}{x^4 + y^2}\, dA$$

by using a coordinate transformation which maps R onto a rectangular region in the new coordinate plane.

4. a. Let S be the boundary of the region bounded by the sphere $x^2 + y^2 + z^2 = 1$ on the top and by the cone $z^2 = x^2 + y^2$ on the sides. Let $\mathbf{n}$ be the outward normal field of S. Evaluate $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ for $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$.

b. Let C be the circle in the plane $2x+2y+z=2$ with center $(0,0,2)$ and radius 3 in the counterclockwise direction as viewed from the origin. Evaluate

$$\oint_C 2y\,dx + 3x\,dy - x\,dz\ .$$

5. a. Find the area of the surface cut from the bottom of the paraboloid $z=x^2+y^2$ by the plane $z=3/4$.

b. Evaluate $\displaystyle\int_0^a \int_0^b e^{f(x,y)}\,dy\,dx$ where a and b are positive numbers and

$$f(x,y) = \begin{cases} b^2x^2 & \text{if } b^2x^2 \geq a^2y^2\,, \\ a^2y^2 & \text{if } b^2x^2 < a^2y^2 \end{cases}$$

6. a. Assume that $\nabla\cdot\mathbf{F} > 0$ for all (x,y) where $\mathbf{F} = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ is a vector field in the plane whose components have continuous partial derivatives. Show that there is <u>no</u> smooth simple closed curve in the plane whose tangent vector is parallel to $\mathbf{F}$ at all its points.

b. Evaluate $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r}$ where

$$\text{curl } \mathbf{F} = \frac{-y}{4x^2+9y^2}\,\mathbf{i} + \frac{x}{4x^2+9y^2}\,\mathbf{j}$$

and C is the unit circle parametrized in the counterclockwise direction.

Spring 1992 Midterm II

1. Let $F(x,y,z) = 2x^3 + 3y^4 + 5z^6 + 7$ where $x = \cos t + \sin t$, $y = \tan t + t^2 + 1$, and $z = 1 - t + 2\ln(2+t)$. Find $\left.\dfrac{dF}{dt}\right|_{t=0}$.

2. Find the distance between the following two lines:

$$L_1 : \begin{cases} x = 3+t \\ y = 2-4t \\ z = t \end{cases} \qquad \text{and} \quad L_2 : \begin{cases} x = 4-s \\ y = 3+s \\ z = -2+3s \end{cases}$$

3. Describe the points $P(\rho,\phi,\theta)$ whose coordinates satisfy $\theta = 3\pi/2$, $\rho = 3\cos\phi$, and sketch.

4. Find the distance from the point$(2,0,3)$ to the plane which is tangent to the surface $4x^2 - y^2 + 4z^2 = 4$ at the point $(1,2,1)$.

5. A function is defined as follows:

$$f(x,y) = \begin{cases} \dfrac{x^3y}{x^5+2y^3} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Show that f is continuous at the origin.

6. A space curve is given by the parametrization $\mathbf{r}(t) = (e^{2t}, t, t^2)$. Find the equation of the osculating plane at the point corresponding to $t = 1$.

Spring 1992 Final

1. Find the points on the paraboloid $z = 4x^2 + 9y^2$ at which the normal line is parallel to the line through $P(-2, 4, 3)$ and $Q(5, -1, 2)$.

2. Do the following series converge or diverge? Give reasons.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{1000^n}$

b. $\displaystyle\sum_{n=5}^{\infty} \frac{n^{1/2}}{(\ln n)^3}$

c. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n^3}$

d. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{n^2}\right)$

3. Find the dimensions of the rectangular box of maximum volume that has three of its faces in the coordinate planes, one vertex at the origin and another vertex in the first octant on the plane $2x + 3y + 5z = 90$.

4. Find the minimum, maximum and the saddle points of the function $f(x,y) = 2x^4 + xy + y^2$.

5. Find the volume of the region bounded by the plane $z = 0$, the cylinder $x^2+y^2 = 4$ and the cylinder $2z = 4 - y^2$.

6. Evaluate the integral

$$\iint \frac{1}{(x^2+y^2)^{5/2}}\, dx\, dy$$

over the region which is bounded by the lines $y = \sqrt{3}x$, $y = x$, $y = 1$ and $y = 2$.

Spring 1991 Midterm I

1. Does the integral $\displaystyle\int_0^\infty \frac{\ln x}{x^2}\,dx$ converge or diverge?

2. a. Find the first three non-vanishing terms in the Taylor series for the function $f(x) = \sin^{-1} x$.

b. Find the radius of convergence of the above series.

3. Let $s_n = 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n}$. Show that $\displaystyle\lim_{n\to\infty}(s_n - \ln n)$ exists.

4. Find the volume of the tetrahedron with vertices at $(2, 1, 1)$, $(1, -1, 2)$, $(0, 1, -1)$ and $(1, -2, 1)$.

5. [*missing*]

6. Find the distance between the point $P(0, 1, 1)$ and the line $x = 1+2t$, $y = -1-t$, $z = 3t$.

Spring 1991 Midterm II

1. Prove that the radius of curvature of a curve parameterized by its arc length is given by $\rho = (\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2)^{-1/2}$ where dot denotes derivative with respect to the arc length.

2. We have 100 vectors in $\mathbf{R}^{100}$ defined as follows:

$$\begin{aligned}
\vec{v}_1 &= (1, 0, 0, \ldots, 0, 0, 1)\\
\vec{v}_2 &= (0, 1, 0, \ldots, 0, 1, 0)\\
&\vdots\\
\vec{v}_{50} &= (0, 0, \ldots, 1, 1, \ldots, 0, 0)\\
\vec{v}_{51} &= (1, 0, 0, \ldots, 0, 0, -1)\\
\vec{v}_{52} &= (0, 1, 0, \ldots, 0, -1, 0)\\
&\vdots\\
\vec{v}_{100} &= (0, 0, \ldots, 1, -1, \ldots, 0, 0)
\end{aligned}$$

Let $\vec{A}$ be the vector $(1, 2, 3, \ldots, 100)$. If $\vec{A} = a_1\vec{v}_1 + \cdots + a_{100}\vec{v}_{100}$ where a_is are real constants, find a_i, $1 \le i \le 100$.

3. Find the equation of the locus of the center of the circle of curvature of the curve $y = x^2$.

4. Let $\vec{u}$ and $\vec{v}$ be two distinct nonzero vectors. Show that the vector $\vec{w} = |\vec{v}|\vec{u} + |\vec{u}|\vec{v}$ bisects the angle between $\vec{u}$ and $\vec{v}$.

5. Find the curvature of the cycloid $x = a(t - \sin t)$, $y = (1 - \cos t)$, $t \ge 0$, at the highest point of the arc.

Spring 1991 Final

1. a. Find the Taylor expansion of $f(x) = c^x$, $c > 0$, around $x = 0$.

b. Test for convergence: $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

2. Find the curvature, the torsion and the normal vector for the space curve

$$\mathbf{R}(t) = a\cos\omega t\mathbf{i} + a\sin\omega t\mathbf{j} + bt\mathbf{k}$$

where a, b, ω are positive constants.

3. Let $f(x, y) = (y - x^2)(y - 2x^2)$.

a. Show that $(0, 0)$ is a critical point of f at which f has a saddle point.

b. Show that on any line through the origin f has a local minimum at $(0, 0)$.

5. Find the volume bounded by the surfaces $z = x^2 + y^2$ and $z = (x^2 + y^2 + 1)/2$.

6. Evaluate

$$\oint_C \left(\frac{y^3}{x} - e^x\right) dx + \left(\frac{y}{x} + e^{y^2}\right) dy$$

where C is the boundary of the region bounded by the curves $y = x^2$, $y = 2x^2$, $y = 1/x$, $y = 3/x$, traced counterclockwise.

Spring 1990 Midterm II

1. Let Π be a plane in 5-dimensional space. Suppose that the vector $\vec{v} = (1, 2, 3, 4, 5)$ is perpendicular to the plane Π and that the point $P(5, 4, 3, 2, 1)$ is on Π. Find the distance of Π to the origin.

2. Find the unit tangent, a unit principal normal and a unit binormal vector along the curve

$$\vec{r}(t) = (3t - t^3)\mathbf{i} + 3t^2\mathbf{j} + (3t + t^3)\mathbf{k} .$$

3. Find the intersection of the xy-plane with the tangent line to the curve

$$\vec{r}(t) = (1 + t)\mathbf{i} - t^2\mathbf{j} + (1 + t^3)\mathbf{k}$$

at $t = 1$.

4. Find $\dfrac{dw}{dt}$ if $w = f(x, y, z)$ and $x = t$, $y = g(t)$, $z = h(t, g(t))$.

5. Let R be the distance from a fixed point $A(a, b, c)$ to any point $P(x, y, z)$. Show that

$$\vec{\nabla} R = \frac{\overrightarrow{AP}}{R} .$$

6. Let $f(x, y) = \phi(x - cy) + \psi(x + cy)$ where c is a constant. Show that

$$c^2 f_{xx} = f_{yy} .$$

Spring 1989 Midterm II

1. Let $c > 0$ be a constant. Find the set of all x for which the following power series converges. Check also the endpoints.

$$\sum_{n=0}^{\infty} c^n x^{2n}$$

2. By using the series find

$$\lim_{x \to 0} \frac{6 \sinh x - 6x - x^3}{x^5} .$$

3. Let $z = f(u, v)$ where f is of classC^2, $u = x^2 + y$, $v = x - 2y^2$. Find $\dfrac{\partial^2 z}{\partial y \partial x}$ in terms of the partial derivatives of z with respect to u and v.

4. Let $p > 0$ be a constant and f be of class C^1. Assume that $\forall t \in \mathbf{R}$ and $\forall (x, y) \in \mathbf{R}^2$ we have

$$f(tx, ty) = t^p f(x, y) .$$

Show that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = pf(x, y) .$$

5. Find and classify all the critical points of

$$f(x, y) = 3x^2y - 9y^3 - x^2 .$$

6. Find the shortest distance from the origin to the surface $xyz^2 = 2$ by using the method of Lagrange multipliers.

7. Find the volume of the solid bounded by $y = 4 - x^2$, $y = x^2 - 4$ and $-y + 4z = 8$.

8. Let $a > 0$ be a constant. Show that

$$\int_0^a \int_0^x f(y)\, dy\, dx = \int_0^a (a - y)f(y)\, dy .$$

Spring 1988 Midterm II

1. Suppose $3 < a_n < 4$ for all n and $\lim_{n\to\infty} a_n = 4$. Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{a_1 a_2 \cdots a_n} .$$

Do not forget to check the end points.

2. Let $a > 0$, $b > 0$ be constants. Find the <u>open</u> interval I of convergence of the power series

$$\sum_{n=1}^{\infty} \left(\frac{a}{n} + bn\right) x^n .$$

For $x \in I$, let $f(x)$ be the sum of the series. Find a closed form for the function $f(x)$.

3. Let U and V be vector spaces over $\mathbf{R}$ with zero elements 0_U and 0_V , respectively. Let $T : U \to V$ be a linear mapping, i.e. $\forall u_1, u_2 \in U, \forall c \in \mathbf{R}$, $T(cu_1) = cT(u_1)$ and $T(u_1 + u_2) = T(u_1) + T(u_2)$. We define

$$\ker(T) = \{u \in U : T(u) = 0_V\} .$$

a. Show that $0_U \in \ker(T)$.

b. Show that $\ker(T)$ is a subspace of U.

4. Let L be a line and P_1 a point in $\mathbf{R}^3$. Show that the distance between P_1 and L is

$$d(P_1, L) = \frac{||\vec{u} \times \overrightarrow{P_0P_1}||}{||\vec{v}||}$$

where $\vec{u}$ is any vector parallel to L and $P_)$ is any point on L.

5. Given the line $L : \dfrac{x-2}{1} = \dfrac{y}{-2}$, $z = 2$ and the point $P_0(1, 2-1)$ find the equation of the plane Π which contains the line L and the point P_0 .

6. Given the plane curve $\mathbf{R}(t) = e^t \cos t\mathbf{i} + e^t \sin t\mathbf{j}$, show that the angle between $\mathbf{R}(t)$ and the acceleration vector $\mathbf{a}(t)$ is constant. Find the angle.

7. For the space curve

$$\mathbf{R}(t) = 3\sin t\mathbf{i} + 3\cos t\mathbf{j} + 4t\mathbf{k}$$

find $\mathbf{T}$, κ, $\mathbf{N}$, $\mathbf{B}$.