## Solution to MATH 112-04 QUIZ 8

**Problem 1**. Write the Maclaurin series for

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \sin(5x) = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1} x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \cos(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad e^{x/4} = \sum_{n=0}^{\infty} \frac{x^n}{4^n n!}$$

**Problem 2**. (a) Find the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{n^2 x^n}{n!}$ and write down the interval of convergence for this series.
(b) Find the sum of the series $\sum_{n=1}^{\infty} \frac{4^n n^2}{n!}$.

**Solution**.
(a) We have $\sum_{n=1}^{\infty} \frac{n^2 x^n}{n!} = \sum_{n=1}^{\infty} c_n x^n$. The radius of convergence $R$ satisfies the equality

$$\frac{1}{R} = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = \lim_{n\to\infty} \frac{(n+1)^2 n!}{(n+1)! n^2} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^2 \frac{1}{n+1} = 0.$$

Thus, the radius of convergence $R = \infty$ and hence, the interval of convergence of the series is $(-\infty, \infty)$.
(b) Consider $f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $-\infty < x < \infty$. Differentiating term-by-term gives

$$\frac{d}{dx}(e^x) = e^x = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d}{dx} x^n = \sum_{n=0}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} \Rightarrow e^x = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!}, -\infty < x < \infty$$

We multiply by $x$ both sides and differentiate term-by-term once more:

$$x e^x = \sum_{n=1}^{\infty} \frac{n x^n}{n!} \Rightarrow \frac{d}{dx}(x e^x) = e^x + x e^x = \sum_{n=1}^{\infty} \frac{n^2 x^{n-1}}{n!}, -\infty < x < \infty$$

We multiply both sides by x and evaluate the series at $x = 4$:

$$x(e^x + x e^x) = \sum_{n=1}^{\infty} \frac{n^2 x^n}{n!} \Rightarrow 4(e^4 + 4e^4) = \sum_{n=1}^{\infty} \frac{4^n n^2}{n!} \Rightarrow \underline{\sum_{n=1}^{\infty} \frac{4^n n^2}{n!} = 20e^4}$$