Solution to MATH 112-04 QUIZ 7

Example 1. Which of the series converge absolutely, converge conditionally, and diverge? State clearly conditions and name of test you are using.

$$(a)\quad \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}} \qquad (b)\quad \sum_{n=1}^{\infty}\frac{\arctan(\sin n)}{n^2} \qquad (c)\quad \sum_{n=0}^{\infty}\cos\left(\frac{\pi n}{2}\right)$$

Solution.
(a) We have $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}} = \sum\limits_{n=1}^{\infty}(-1)^n u_n$, where $u_n = \frac{1}{\sqrt{n}}$. Since 1) $u_n > 0 \quad \forall n \geq 1$, 2) $u_n \geq u_{n+1} \quad \forall n \geq 1$ (Indeed, $\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}} \quad \forall n \geq 1$), 3) $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty}\frac{1}{\sqrt{n}} = 0$, then, by the Alternating Series Test, series $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converges.
The series of absolute values $\sum\limits_{n=1}^{\infty}\left|\frac{(-1)^n}{\sqrt{n}}\right| = \sum\limits_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges as p-series with $p = \frac{1}{2} < 1$. Therefore, $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converges conditionally.

(b) We have $\sum\limits_{n=1}^{\infty}\frac{\arctan(\sin n)}{n^2} = \sum\limits_{n=1}^{\infty} a_n$, where $a_n = \frac{\arctan(\sin n)}{n^2}$. Since 1) $|a_n| \leq \frac{\pi}{2n^2}$ and 2) $\sum\limits_{n=1}^{\infty}\frac{\pi}{2n^2}$ converges as p-series with $p = 2 > 1$, then by the Direct Comparison Test, $\sum_{n=1}^{\infty}|a_n|$ converges and, by the Absolute series Test, series $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty}\frac{\arctan(\sin n)}{n^2}$ converges, and hence converges absolutely.

(c) $\sum\limits_{n=0}^{\infty}\cos\left(\frac{\pi n}{2}\right) = \sum\limits_{n=0}^{\infty} a_n$, where $a_n = \cos\left(\frac{\pi n}{2}\right)$. We have

$$a_{4n} = 1, \quad a_{4n+1} = 0, \quad a_{4n+2} = -1, \quad a_{4n+3} = 0,$$

that implies that $\lim\limits_{n\to\infty} a_n$ does not exist. Therefore, by the n-th term test for divergence, series $\sum\limits_{n=0}^{\infty}\cos\left(\frac{\pi n}{2}\right)$ diverges.