Solution to MATH 112-04 QUIZ 6

Problem 1. Determine whether each of the following series is convergent or divergent. State clearly the name and the conditions of the test you are using.

$$(a) \quad \sum_{n=1}^{\infty} e^{-n}n^3, \qquad (b) \quad \sum_{n=2}^{\infty} \frac{1}{n\sqrt{n-\sin n}}, \qquad (c) \quad \sum_{n=1}^{\infty} \frac{\sqrt[n]{3}}{n\ln\left(1+\frac{1}{n}\right)}.$$

Solution.

(a) We have $\sum\limits_{n=1}^{\infty} e^{-n}n^3 = \sum\limits_{n=1}^{\infty} a_n$, where $a_n = e^{-n}n^3$. Since 1) $a_n > 0, \forall n \geq 1$ and

$$2) \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{e^{-(n+1)}(n+1)^3}{e^{-n}n^3} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^3}{e} = \frac{1}{e} < 1,$$

then, by the <u>Ratio Test</u>, series $\sum\limits_{n=1}^{\infty} e^{-n}n^3$ <u>converges</u>.

(b) We have $\sum\limits_{n=2}^{\infty} \frac{1}{n\sqrt{n-\sin n}} = \sum\limits_{n=2}^{\infty} a_n$, where $a_n = \dfrac{1}{n\sqrt{n-\sin n}} = \dfrac{1}{n^{3/2}\sqrt{1-\frac{\sin n}{n}}}$/ As $n \to infty$ a_n behaves like $b_n = \frac{1}{n^{3/2}}$. Since 1) $a_n > 0, b_n > 0 \forall n \geq 2$,

$$2) \quad \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^{3/2}\sqrt{1-\frac{\sin n}{n}}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{1}{\sqrt{1-\frac{\sin n}{n}}} = 1, \quad 0 < 1 < \infty,$$

3) $\sum\limits_{n=2}^{\infty} b_n = \sum\limits_{n=2}^{\infty} \frac{1}{n^{3/2}}$ converges as p-series with $p = \frac{3}{2} > 1$, then, by the <u>Limit Comparison Test</u>, series $\sum\limits_{n=2}^{\infty} \dfrac{1}{n\sqrt{n-\sin n}}$ <u>converges</u>.

(c) We have $\sum\limits_{n=1}^{\infty} \dfrac{\sqrt[n]{3}}{n\ln\left(1+\frac{1}{n}\right)} = \sum\limits_{n=1}^{\infty} a_n$, where $a_n = \dfrac{\sqrt[n]{3}}{\ln\left(1+\frac{1}{n}\right)^n}$. Since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sqrt[n]{3}}{\ln\left(1+\frac{1}{n}\right)^n} = \frac{1}{\ln e} = 1 \neq 0,$$

then, by the <u>n-th term test for divergence</u>, series $\sum\limits_{n=1}^{\infty} \dfrac{\sqrt[n]{3}}{n\ln\left(1+\frac{1}{n}\right)}$ <u>diverges</u>.