MATH 112-04 QUIZ 5

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Example 1. Determine whether $\int\limits_0^\infty \dfrac{dx}{(x^3+\arctan x)^{1/2}}$ converges.

Solution. We have $\int\limits_0^\infty \frac{dx}{(x^3+\arctan x)^{1/2}} = \int\limits_0^1 \frac{dx}{(x^3+\arctan x)^{1/2}} + \int\limits_1^\infty \frac{dx}{(x^3+\arctan x)^{1/2}} = I_1 + I_2$.
<u>Step 1: Investigate whether I_1 converges.</u> $\frac{1}{\sqrt{x^3+\arctan x}}$ is a continuous function on $(0,1]$ and

$$\lim_{x\to 0} \frac{\frac{1}{(x^3+\arctan x)^{1/2}}}{\frac{1}{x^{1/2}}} = \lim_{x\to 0} \frac{1}{(x^2+\frac{\arctan x}{x})^{1/2}} = 1, \qquad (*)$$

since by H'Lopital's Rule $\lim\limits_{x\to 0} \frac{\arctan x}{x} = \lim\limits_{x\to 0} \frac{\frac{1}{1+x^2}}{1} = 1$. Since $\int\limits_0^1 \frac{dx}{x^{1/2}}$ converges and (*) holds, then by the Limit Comparison Test(LCT), $I_1 = \int\limits_0^1 \dfrac{dx}{(x^3+\arctan x)^{1/2}}$ converges.

<u>Step 2: Investigate whether I_2 converges.</u> On $[1,\infty)$, $\sqrt{x^3+\arctan x} > \sqrt{x^3}$, so $\frac{1}{(x^3+\arctan x)^{1/2}} < \frac{1}{x^{3/2}}$ for any x on $[1,\infty)$.
Since $\int\limits_1^\infty \frac{dx}{x^{3/2}}$ converges and $\frac{1}{(x^3+\arctan x)^{1/2}} < \frac{1}{x^{3/2}}$ for any x on $[1,\infty)$, then by the Direct Comparison Test(DCT), $I_2 = \int\limits_1^\infty \dfrac{dx}{(x^3+\arctan x)^{1/2}}$ converges.

Therefore, $\int\limits_0^\infty \dfrac{dx}{(x^3+\arctan x)^{1/2}} = I_1 + I_2$ converges.

Problem 1. Determine whether $\int\limits_0^\infty \dfrac{dx}{(\sin^2 x + x^4)^{1/3}(1+e^x)}$ converges.

Solution. We have $\int\limits_0^\infty \frac{dx}{(\sin^2 x+x^4)^{1/3}(1+e^x)} = \int\limits_0^1 \frac{dx}{(\sin^2 x+x^4)^{1/3}(1+e^x)} + \int\limits_1^\infty \frac{dx}{(\sin^2 x+x^4)^{1/3}(1+e^x)} = I_1 + I_2$.
<u>Step 1: Investigate whether I_1 converges.</u> $\frac{1}{(\sin^2 x+x^4)^{1/3}(1+e^x)}$ is a continuous function on $(0,1]$ and

$$\lim_{x\to 0} \frac{\frac{1}{(\sin^2 x+x^4)^{1/3}(1+e^x)}}{\frac{1}{x^{2/3}}} = \lim_{x\to 0} \frac{1}{(x^2+\left(\frac{\sin x}{x}\right)^2)^{2/3}(1+e^x)} = \frac{1}{2}, \qquad (**)$$

since $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$. Since $\int\limits_0^1 \frac{dx}{x^{2/3}}$ converges and (**) holds, then by the Limit Comparison Test(LCT), $I_1 = \int\limits_0^1 \dfrac{dx}{(\sin^2 x + x^4)^{1/3}(1+e^x)}$ converges.
<u>Step 2: Investigate whether I_2 converges.</u> On $[1,\infty)$, $\sin^2 x + x^4 \geq x^4 \geq 1$ and $1+e^x > e^x$,

so $\frac{1}{(\sin^2 x+x^4)^{1/3}(1+e^x)} < \frac{1}{e^x}$ for any x on $[1,\infty)$.

Since $\int\limits_1^\infty \frac{dx}{e^x} = -e^{-x}|_0^\infty = 1$ converges and $\frac{1}{(\sin^2 x+x^4)^{1/3}(1+e^x)} < \frac{1}{e^x}$ for any x on $[1,\infty)$, then by the Direct Comparison Test(DCT), $I_2 = \int\limits_1^\infty \frac{dx}{(\sin^2 x + x^4)^{1/3}(1+e^x)}$ converges.

Therefore, $\int\limits_0^\infty \frac{dx}{(\sin^2 x + x^4)^{1/3}(1+e^x)} = I_1 + I_2$ converges.

Hint: You may use that $\lim x \to 0 \frac{\sin x}{x} = 1$ *and* $\int\limits_0^\infty \frac{dx}{e^x}$ *converges*