Solution to MATH 112-04 QUIZ 3

Problem 1. Evaluate the following limits

$$(a) \qquad \lim_{x\to 1^+}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right), \qquad\qquad (b) \qquad \lim_{x\to+\infty}(\ln x)^{1/(\ln x)^2}.$$

Solution.

$$(a) \qquad \lim_{x\to 1^+}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right)=\lim_{x\to 1^+}\frac{\ln x-(x-1)}{(x-1)\ln x}=\lim_{x\to 1^+}\frac{\frac{1}{x}-1}{\ln x+\frac{x-1}{x}}=$$

$$=\lim_{x\to 1^+}\frac{1-x}{x\ln x+(x-1)}=\lim_{x\to 1^+}\frac{-1}{\ln x+x\frac{1}{x}+1}=-\frac{1}{2}.$$

In part (a) we have used H'Lopital's Rule twice.

$$(b) \qquad \lim_{x\to+\infty}(\ln x)^{1/(\ln x)^2}=\lim_{x\to+\infty}e^{\frac{\ln(\ln x)}{(\ln x)^2}}=e^{\lim_{x\to+\infty}\frac{\ln(\ln x)}{(\ln x)^2}}=$$

$$=e^{\lim_{x\to+\infty}\frac{1}{(\ln x)x}\frac{x}{2\ln x}}=e^{\lim\limits_{x\to+\infty}\frac{1}{2(\ln x)^2}}=e^0=1.$$

In part (b) we have used H'Lopital's Rule once.

Problem 2. Let $g(x)=\int\limits_{-1}^{x}\frac{3dt}{4t^2+4t+4}$ and $f(x)=\sec^{-1}(2g(x))$. Find $g(0)$ and $f'(0)$.

Solution. We have

$$1) \quad g(x)=\int\limits_{-1}^{x}\frac{3dt}{4t^2+4t+4}=\int\limits_{-1}^{x}\frac{3dt}{(2t+1)^2+3}=\frac{1}{2}\int\limits_{-1}^{2x+1}\frac{3du}{u^2+3}=\frac{3}{2}\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)\Big]_{-1}^{2x+1}=$$

$$=\frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+\frac{\pi\sqrt{3}}{12} \qquad \Rightarrow \qquad g(0)=\frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)+\frac{\pi\sqrt{3}}{12}=\frac{\pi\sqrt{3}}{6},$$

$$2) \quad f'(x)=\frac{1}{|2g(x)|\sqrt{(2g(x))^2-1}}\frac{d}{dx}(2g(x))=\frac{2g'(x)}{|2g(x)|\sqrt{(2g(x))^2-1}}$$

By the First Fundamental Theorem,

$$g'(x)=\frac{d}{dx}\int\limits_{-1}^{x}\frac{3dt}{4t^2+4t+4}=\frac{3}{4x^2+4x+4}.$$

Therefore, $g'(0)=\frac{3}{4}$ and

$$f'(0)=\frac{g'(0)}{|g(0)|\sqrt{4(g(0))^2-1}}=\frac{\frac{3}{4}}{\frac{\pi}{2\sqrt{3}}\sqrt{\frac{4\pi^2}{12}-1}}=\frac{9}{2\pi\sqrt{\pi^2-3}}$$