Solution to MATH 112-04 QUIZ 2

Problem 1. Evaluate the following integrals

$$(a) \qquad \int \frac{2+\tan(\ln(2x))}{x}dx \qquad (b) \qquad \int_0^{\ln 10} e^t\sqrt{e^t-1}dt.$$

Solution.

$$(a) \qquad \int \frac{2+\tan(\ln(2x))}{x}dx = \int \frac{2}{x}dx + \int \frac{\tan(\ln(x))}{x}dx \underset{\ln(2x)=u,\ \frac{1}{x}dx=du}{=} 2\ln|x| + \int \tan u du =$$

$$= 2\ln|x| + \int \frac{\sin u}{\cos u}du = 2\ln|x| - \int \frac{d(\cos u)}{\cos u} = 2\ln|x| - \ln|\cos(\ln(2x))| + C.$$

$$(b) \qquad \int_0^{\ln 10} e^t\sqrt{e^t-1}dt \underset{e^t=u,\ e^t dt=du}{=} \int_1^{10} \sqrt{u-1}du = \frac{2}{3}(u-1)^{3/2}\Big]_1^{10} = 18.$$

Problem 2. Let $f(x) = (\ln x)^{\frac{1}{\ln x}}$.
(a) Find the domain of $f(x)$ and find $\frac{df}{dx}$.
(b) Find point $x = a$ at which function $f(x)$ takes its maximum value. What is $f(a)$?

Solution.

$(a) f(x) = (\ln x)^{\frac{1}{\ln x}} = e^{\frac{\ln(\ln x)}{\ln x}} \Longrightarrow$ the domain of $f(x)$ is all x such that $\ln x > 0 \Longleftrightarrow x > 1$.

We have

$$y = (\ln x)^{\frac{1}{\ln x}} \Longrightarrow \ln y = \frac{\ln(\ln x)}{\ln x} \Longrightarrow \frac{1}{y}\frac{dy}{dx} = -\frac{1}{(\ln x)^2}\frac{1}{x}\ln(\ln x) + \frac{1}{\ln x}\frac{1}{\ln x}\frac{1}{x} = \frac{1-\ln(\ln x)}{x(\ln x)^2}$$

$$\Longrightarrow \frac{dy}{dx} = y\frac{1-\ln(\ln x)}{x(\ln x)^2} \qquad \Longrightarrow \qquad \frac{df}{dx} = \frac{(\ln x)^{\frac{1}{\ln x}}(1-\ln(\ln x))}{x(\ln x)^2}, \quad x > 1.$$

(b) $f(x)$ is defined for $x > 1$. Since $(\ln x)^{\frac{1}{\ln x}}$, x, $(\ln x)^2$ take positive values for $x > 1$ then
1) $\frac{df}{dx} > 0$ if and only if $1 - \ln(\ln x) > 0 \Longleftrightarrow \ln(\ln x) < 1 \Longleftrightarrow \ln x < e \Longleftrightarrow x < e^e$
and
2) $\frac{df}{dx} < 0$ if and only if $1 - \ln(\ln x) < 0 \Longleftrightarrow \ln(\ln x) > 1 \Longleftrightarrow \ln x > e \Longleftrightarrow x > e^e$
Therefore, on $(1, e^e)$ $f(x)$ is an increasing function and on (e^e, ∞) $f(x)$ is a decreasing function $\Longrightarrow x = e^e$ is a point where $f(x)$ takes its maximum value.

$$f(e^e) = (\ln e^e)^{\frac{1}{\ln e^e}} = e^{\frac{1}{e}} \longleftarrow \text{maximum value}$$