Solution to MATH 112-04 QUIZ 1

Problem 1. Let

$$f(x) = \frac{2x+1}{x+3} = 2 - \frac{5}{x+3}, \qquad x < -3.$$

(a) (no points given) Sketch the graph of function $y = f(x)$.
(b) Find the inverse of function $f(x)$. Do not forget to indicate the domain of $f^{-1}(x)$.

Solution. (b) $y = 2 - \frac{5}{x+3}$, $x < -3$, $\Rightarrow$ $\frac{5}{x+3} = 2 - y$, $x + 3 < 0$, $\Rightarrow$ $x + 3 = \frac{5}{2-y}$, $2 - y < 0$, $\Rightarrow$ $x = -3 + \frac{5}{2-y} = \frac{3y-1}{2-y}$, $y > 2$, $\Rightarrow$ $x = f^{-1}(y) = \frac{3y-1}{2-y}$, $y > 2$, $\Rightarrow$ $y = f^{-1}(x) = \frac{3x-1}{2-x}$, $x > 2$.
Answer: $f^{-1}(x) = \frac{3x-1}{2-x}$, $x > 2$.

Problem 2. Let $f(x) = x^3 + x$.
(a) (no points given) Explain why $f(x)$ is one-to-one on the whole real line.
(b) Without finding the formula for $f^{-1}(x)$ evaluate $\frac{df^{-1}}{dx}$ at points $x = 2$ and $x = -10$.

Solution. (a) We have $f'(x) = 3x^2 + 1 > 0$ for all $x \in (-\infty, \infty)$. It implies that $f(x)$ is a strictly increasing function on the whole real line. Therefore, $f(x)$ is a one-to-one function on the whole real line.
(b) We have $\left.\frac{df^{-1}}{dx}\right|_{x=f(a)} = \frac{1}{\frac{df}{dx}\big|_{x=a}} = \frac{1}{(3x^2+1)\big|_{x=a}}$
(i) For $x = 2 = f(a) = a^3 + a$ there is only one solution $a = 1$ (only one is because we have proved in part (a) that $f(x)$ is a one-to-one function). Therefore,

$$\left.\frac{df^{-1}}{dx}\right|_{x=2} = \frac{1}{\frac{df}{dx}\big|_{x=1}} = \frac{1}{(3x^2+1)\big|_{x=1}} = \frac{1}{4}$$

(ii) For $x = -10 = f(a) = a^3 + a$ there is only one solution $a = -2$ Therefore,

$$\left.\frac{df^{-1}}{dx}\right|_{x=-10} = \frac{1}{\frac{df}{dx}\big|_{x=-2}} = \frac{1}{(3x^2+1)\big|_{x=-2}} = \frac{1}{13}$$

Answer:

$$\left.\frac{df^{-1}}{dx}\right|_{x=2} = \frac{1}{4}$$

$$\left.\frac{df^{-1}}{dx}\right|_{x=-10} = \frac{1}{13}$$