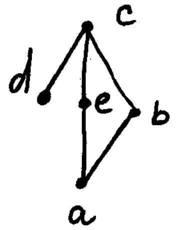
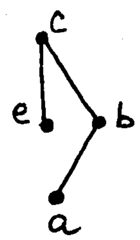
Subgraphs, Complements

<u>Definition</u>. Let $G=(V,E)$ be a graph (directed or undirected). Then $G_1=(V_1,E_1)$ is called a <u>subgraph of G</u> if V_1 is a nonempty subset of V, E_1 is a nonempty subset of E and each edge in E_1 is incident with vertices in V_1.

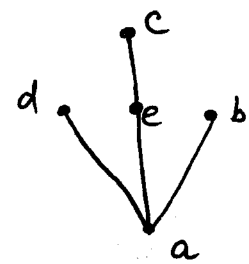
Examples



$G=(V,E)$
$V=\{a,b,c,d,e\}$



$G_1=(V_1,E_1)$ is a subgraph of G
$V_1=\{a,b,c,e\}$

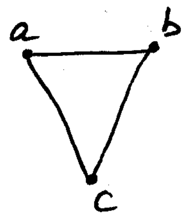


$G_2=(V_2,E_2)$
$V_2=\{a,b,c,d\}$
is <u>not</u> a subgraph of G.

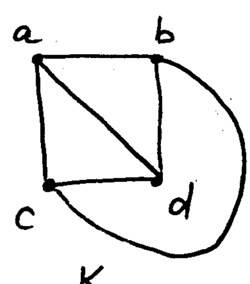
<u>Definition</u>. Let V be a set of n vertices. <u>The complete graph</u> on V, denoted K_n, is a loop-free undirected graph where for all $a,b\in V$, $a\neq b$, there is an edge $\{a,b\}$.

Examples.

K_2



K_3



K_4

<u>Remark</u>: Let $K_n=(V,E_n)$. Then $|E_n|=\frac{n(n-1)}{2}$.

<u>Definition</u>. Let G be a loop-free undirected graph on n vertices. <u>The complement of G</u>, denoted $\overline{G}$, is the subgraph of K_n consisting of the n vertices

in G and all edges that are not in G.

Remark. Let $G=(V,E)$, $|V|=n$. Then, if $K_n=(V,E_n)$, $\overline{G}=(V, E_n\setminus E)$.

Example 1. Let G be an undirected graph with n vertices. If the number of edges in G is equal to the number of edges in $\overline{G}$, how many edges G has?

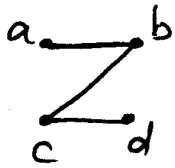
Solution.

Let $G=(V,E)$, $|V|=n$, $|E|=e$. Then $\overline{G}=(V, E_n\setminus E)$. Therefore, $|E_n\setminus E| = \frac{n(n-1)}{2} - e$. We have,

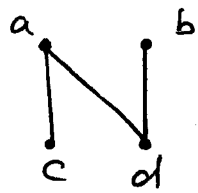
$$e = \frac{n(n-1)}{2} - e, \text{ i.e. } 2e = \frac{n(n-1)}{2}, \text{ i.e. } e = \frac{n(n-1)}{4}.$$

Example 2. Give an example of a graph on 4 vertices s.t. the number of edges in G is the same as the number of edges in $\overline{G}$.

Solution.



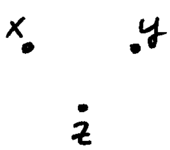
$G=(V,E)$



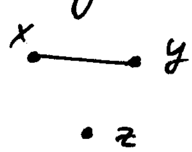
$\overline{G}$

Example 3. How many subgraphs $H=(V,E)$ of K_6 satisfy $|V|=3$?

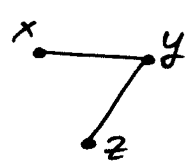
Solution. We can choose 3 vertices out of 6 in $C(6,3)=\frac{6!}{3!3!}=20$ different ways. Let us choose x, y, z vertices.



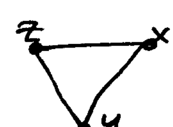
0 edges
1 possibility



1 edge
3 possibilities



2 edges
3 possibilities



3 edges
1 possibilit

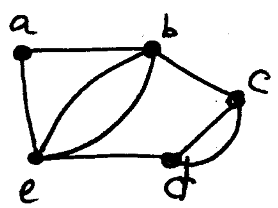
Totally there are $20\times(1+3+3+1)=160$ different subgraphs of K_6 with 3 vertices.

Multigraphs.

<u>Definition</u>. Let V be a finite nonempty set. We say that the pair (V,E) determines a <u>multigraph</u> G with vertex set V and edge set E if, for some $x,y\in V$, there are two or more edges in E of the form

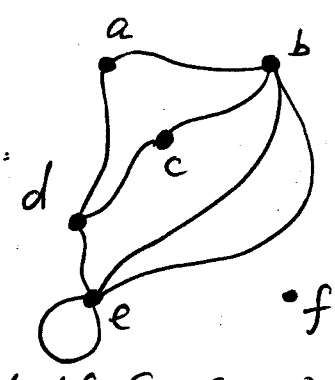
(a) (x,y) (for a directed multigraph), or
(b) $\{x,y\}$ (for an undirected multigraph).

<u>Example</u>.



<u>Definition</u>. Let G be an undirected graph or multigraph. For each vertex v of G, the degree of v, $\deg(v)$, is the number of edges in G that are incident with v. A loop at a vertex v is considered as two incident edges for v.

<u>Example</u>.



$\deg(e)=5$
$\deg(d)=3$
$\deg(a)=2$
$\deg(f)=0$.

<u>Remark 1</u>. If $G=(V,E)$ is an undirected graph or multigraph, then $\sum_{v\in V}\deg(v)=2|E|$.

Remark 2. For any undirected graph or multigraph, the number of vertices of odd degree must be even.

Example. Determine $|V|$ for the following graphs or multigraphs

(a) G has 9 edges and all vertices have degree 3.

(b) G has 10 edges with two vertices of degree 4 and all others of degree 3.

Solution: By Remark 1,

(a) $3|V| = 2 \cdot |E| = 2 \cdot 9 = 18 \Rightarrow |V| = 6.$

(b) $4 \cdot 2 + 3(|V| - 2) = 2 \cdot |E| = 2 \cdot 10 = 20 \Rightarrow |V| = 6.$

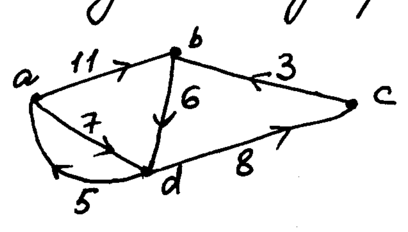
Example. Is it possible to have a graph with 15 edges and such that all vertices have degree 4.

Solution. If such graph $G = (V, E)$ exists then $4|V| = 2|E| = 2 \cdot 15 = 30$. Since 30 is not divisible by 4 then such graph does not exist.

Weighted graphs.

Definition. Let $G = (V, E)$ be a loop-free connected directed graph. To each edge $e = (a, b)$ we assign a positive real number called the weight of e, denoted by $wt(e)$, or $wt(a, b)$. If $x, y \in V$ but $(x, y) \notin E$ we define $wt(x, y) = \infty$.

Graph $G = (V, E)$, where for each edge the number is assigned (as the weight of this edge) is called a weighted graph.



$wt(a, b) = 11,$ $\quad wt(b, a) = \infty,$

$wt(c, b) = 3,$ $\quad wt(a, c) = \infty.$

$wt(d, a) = 5,$

$wt(a, d) = 7,$

Let $G=(V,E)$ be a directed weighted graph
For each $e=(x,y)\in E$, $wt(e)$ may represent the lenght of a road from a to b, the time it takes to travel on this road from a to b, the cost of traveling from a to b on this road.

Definition. For $a,b\in V$, suppose that $v_1, v_2, \dots, v_n \in V$ and that the edges $(a,v_1), (v_1,v_2), \dots, (v_n, b)$ provide a directed path in $G=(V,E)$ from a to b. The length of this path is defined as

$$wt(a,v_1)+wt(v_1,v_2)+\dots+wt(v_n,b).$$

The length of a shortest directed path in G from a to b is called the (shortest) distance from a to b and denoted by $d(a,b)$.

Agreement: 1) $\forall a\in V$ $d(a,a)=0$

2) if there is no path in G from a to b then, we define $d(a,b)=\infty$.

Properties of $d(a,b)$: Let $v_0\in V$, $S\subset V$

Define the distance from v_0 to $\bar{S}$ by

$$d(v_0,\bar{S}) = \min_{v\in \bar{S}} \{d(v_0,v)\}$$

If $d(v_0,\bar{S})<\infty$ then $\exists\, v_{m+1}\in \bar{S}$ s.t. $d(v_0,\bar{S})=d(v_0,v_{m+1})$.
Here $P: (v_0,v_1), (v_1,v_2), \dots, (v_m, v_{m+1})$ is a shortest directed path in G from v_0 to v_{m+1}. Let us show that

1) $v_0, v_1, v_2, \dots, v_m \in S$ and

2) $P': (v_0,v_1), (v_1,v_2), \dots, (v_{k-1}, v_k)$ is a shortest directed path from v_0 to v_k, for each $1\le k\le m$.

Proof of 1): Assume that $\exists\, v_i$, $1\le i\le m$ s.t. $v_i\in\bar{S}$. Then $P'': (v_0,v_1), (v_1,v_2), \dots, (v_{i-1}, v_i)$ is a path from v_0 to an element in $\bar{S}$.

Therefore, $d(v_0, \bar{S}) \le wt(v_0, v_1) + \ldots + wt(v_{i-1}, v_i) <$

$$< wt(v_0, v_1) + \ldots + wt(v_{i-1}, v_i) + wt(v_i, v_{i+1}) + \ldots + wt(v_m, v_{m+1})$$

It contradicts the definition of $d(v_0, \bar{S})$. So, our assumption was wrong. Hence, among $v_0, v_1, \ldots, v_m$ we do not have an element from $\bar{S}$.

Proof of 2): Assume that $d(v_0, v_k) < wt(v_0, v_1) + \ldots + wt(v_{k-1}, v_k)$. Then $d(v_0, \bar{S}) \le d(v_0, v_k) + wt(v_k, v_{k+1}) + \ldots + wt(v_m, v_{m+1})$

$$< wt(v_0, v_1) + wt(v_1, v_2) + \ldots + wt(v_m, v_{m+1}).$$

It contradicts to the definition of P. Hence, our assumption was wrong. Therefore, P' is a shortest directed path from ~~a~~ v_0 to v_k.

We have, from (1) and (2),

$$d(v_0, \bar{S}) = \min\{d(v_0, u) + wt(u, w)\},$$

where minimum is evaluated over all $u \in S$, $w \in \bar{S}$. If a minimum occurs for $u = x$ and $w = y$ then

$$d(v_0, y) = d(v_0, x) + wt(x, y)$$

is the shortest distance from v_0 to y.

Problem. Let $G = (V, E)$ be a weighted graph with $|V| = n$. Let v_0 be a fixed vertex. Find the shortest distance from v_0 to all other vertices in G.

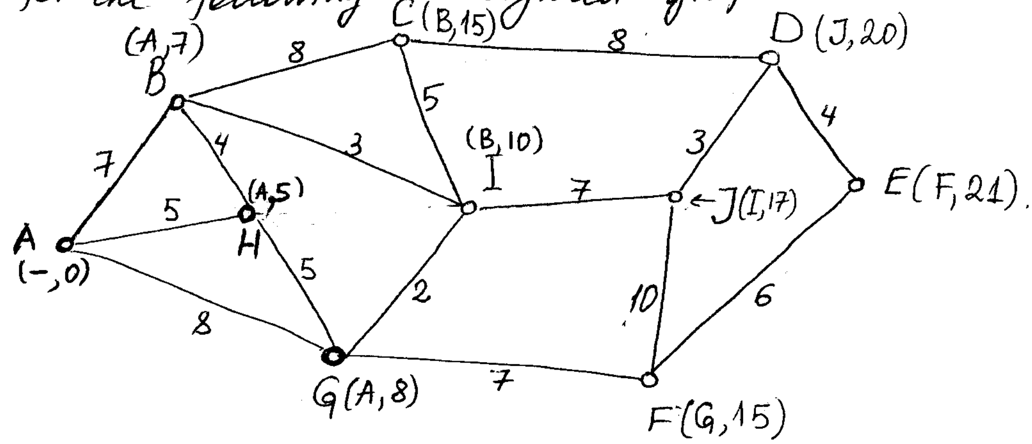
To solve this problem, follow the following procedure (discovered by Dijkstra)

Step 1. Assign to v_0 the label $(-, 0)$.

Step 2. (a) For each labeled vertex $u(x, d)$ and for each unlabeled vertex v adjacent to u (there is an edge (u, v)), compute $d + wt(u, v)$.

(b) For each labeled vertex $u(x, d)$ and unlabeled adjacent vertex v giving minimum $d' = d + w(u, v)$, assign to v the label (u, d'). If a vertex can be labeled (x, d') for various vertices x, make any choice.

Let us find distances from A to all other vertices for the following weighted graph.



First, give A the label $(-,0)$. There are three edges incident with A with weights 7, 5, 8. Since $d=0$, vertex H gives the smallest value $d+wt\{A,H\}$, so H acquires the label $(A,5)$.
Now we repeat Step 2 for the two vertices labeled so far.
There are two unlabeled vertices adjacent to the vertex A. The numbers $d+wt\{e\}$ are $0+7=7$ and $0+8=8$. There are also two unlabeled vertices adjacent to the other labeled vertex H, and for these $d+wt\{e\}$ are $5+4=9$ and $5+5=10$.
The smallest $d+wt\{e\}$ is 7 corresponding to the labeled vertex A and the unlabeled $v=B$. Thus, B is labeled $(A,7)$. Again we repeat step 2.
Now there are three labeled vertices:

A $\to$ one adjacent unlabeled vertex G : $d+wt\{e\}=0+8=8$

B $\to$ adjacent unlabeled vertex C: $d+wt\{e\}=7+8=15$
vertex I: $d+wt\{e\}=7+3=1$

H $\to$ one adjacent unlabeled vertex G: $d+wt\{e\}=5+5=10$.

The smallest $d+wt\{e\}$ is 8, corresponding to edge G. So G acquires the label $(A,8)$

We repeat step 2. There are 4 labeled vertices A, B, H, G.
All vertices adjacent to A, H are already labeled. We looked only for B and G:

B $\to$ two adj. unl. vertices C : $d+wt\{e\}=15$
I : $d+wt\{e\}=10$

~~H~~G $\to$ two adj unl. vertices I : $d+wt\{e\}=8+2=10$
F : $d+wt\{e\}=8+7=15$

The minimum $d+wt\{e\}$ occurs only with I and either edge $\{B,I\}$ or $\{G,I\}$. We can therefore assign to I either the label $(B,10)$ or $(G,10)$. We opt for $(B,10)$.

Repeating Step 2, we have to look only for vertices B, G, I:

B → adj. unl. C → d+w{e} = 7+8 = 15

G → adj. unl. F → d+w{e} = 7+8 = 15

I → adj. unl. C → d+w{e} = 10+5 = 15

J → d+w{e} = 10+7 = 17

We have to label two vertices C and F. We assign labels (B, 15) for C and (G, 15) for F.

We have to look now at vertices C, I, F:

C → adj. unl. D → d+w{e} = 15+8 = 23

I → adj. unl. J → d+w{e} = 10+7 = 17

F → adj. unl. J → d+w{e} = 15+10 = 25

E → d+w{e} = 15+6 = 21

We label J with (I, 17).

Next, we look at C, F, J:

C → adj. unl. D → d+w{e} = 23

F → adj. unl. E → d+w{e} = 21

J → adj. unl. D → d+w{e} = 17+3 = 20

We label D with (J, 20).

Finally, consider D, F:

D → adj. unl. E → d+w{e} = 20+4 = 24

F → adj. unl. E → d+w{e} = 15+6 = 21

We label E with (F, 21)

Since E was labeled last, the algorithm has actually found the length of a shortest route from A to any vertex.

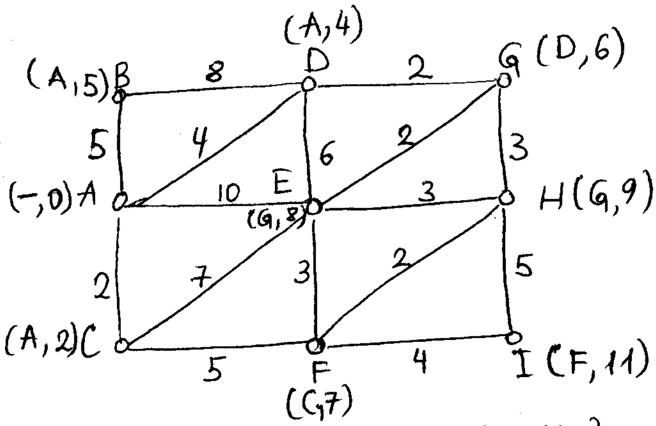
For example,

A → B → I → J is a shortest path to J of length 17.

A → G → F → E is a shortest path to E, of length 21.

Week 9

Problem 1. Apply The Traveling Salesman's Procedure to find the length of the shortest path from A to every other vertex. Show the final labels on all vertices. Also find the shortest path from A to H.



(1) A → adj. unl. B → 5 ($d+wt\{e\}$)
D → 4
C → 2
E → 10

Label C with (A, 2)

(2) A → adj. unl. B → 5 ($d+wt\{e\}$)
D → 4
E → 10

C → adj. unl. E → 2+7=9
F → 2+5=7

Label D with (A,4)

(3) A → adj. unl. B → 5 ($d+wt\{e\}$)
E → 10

D → adj. unl. G → 6
E → 10
B → 12

C → adj. unl. E → 9
F → 7

Label B with (A,5)

(4) We have to look at A, B, C, D.
B does not have unl. adj. vertices.
We are rest with A, C, D, i.e.
see (3) without edge {A, B}. Then
we have to lavel G with (D, 6)

(5) A → adj. unl. E → 10
D → adj. unl. E → 10
C → adj. unl. E → 9
F → 7
G → adj. unl. E → 8
H → 9

Label F with (C,7)

(6) adj. unl. $d+wt\{e\}$
A → E → 10
D → E → 10
C → E → 9
G → E → 8
H → 9
F → E → 10
H → 9
I → 11

Label E with (G,8)

(7) unl. $d+wt\{e\}$
G → H → 9
F → H → 9
I → 11

E → H → 11

Label H with (G,9)
or (F,9)

(8) unl. $d+wt\{e\}$
F → I → 11

H → I → 14

Label I with (F, 11)

Answer: The shortest path from A to H is
A → D → G → H, of length 9.