Generalization of the Pigeonhole Principle.

Theorem. If the number of pigeons is more than k times the number of pigeonholes, then some pigeonhole must contain at least $k+1$ pigeons.

Proof. Suppose that there are p pigeonholes and q pigeons. If no pigeonhole contains at least $k+1$ pigeons, then each of the p pigeonholes contains at most k pigeons. So the total number of pigeons cannot exceed kp. Thus, if the number of pigeons is more than $k \times$ (the number of pigeonholes), i.e. $q > kp$, then some pigeonhole must contain at least $k+1$ pigeons. ∎

Example 1. A conference room contains 8 tables and 105 chairs. What is the smallest possible number of chairs at the table having the most seats?

Solution: We have $p = 8$ holes (tables) to put $q = 105$ pigeons (chairs) in. We know that if $q > k \cdot p$ then some pigeonhole must contain $\geq k+1$ pigeons. Since $105 = q > 8 \cdot \underline{\underline{13}} = 8\underline{\underline{k}}$, then there must be a table with $\geq k+1 = 14$ chairs around it. This number 14 is the smallest possible number of chairs at the table having the most seats.

Example 2. How many books must be chosen from among 24 mathematics books, 25 computer science books, 21 literature books and 15 economics books in order to assure that there are at least 12 books of the same subject.

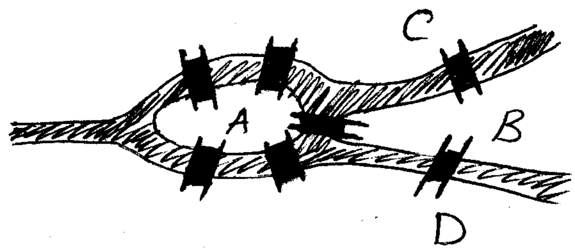
Solution: We have 4 holes to put books in: Hole 1 for math books, Hole 2 for CS books, Hole 3 for Lit. books, Hole 4 for ECON. books. If the number of pigeons (books), q, is more then $4 \cdot 11$, then at least one hole must contain $11+1$ books. Therefore, we must choose ≥ 45 books. Then at least 12 will be of the same subject.

Graphs

There are many concrete practical problems that can be simplified and solved by looking at them from a different point of view.

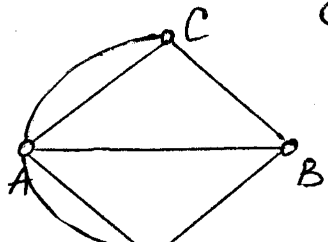
<u>The Königsberg Bridge Problem</u> was a long-standing problem until it was solved in 1736 by the great Swiss mathematician Léonhard Euler (1707–1783). Euler spent the last 17 years of his life blind but still very active.

In the eighteen century, Königsberg was the capital of East Prussia. Pregel River flowed through town and split into two branches around Kneiphof island. Seven bridges crossed the river, providing links among the four land masses A, B, C, D.



People wondered if it were possible to start on one of the land masses, walk over each of the seven bridges exactly once, and return to the starting point (without getting wet)

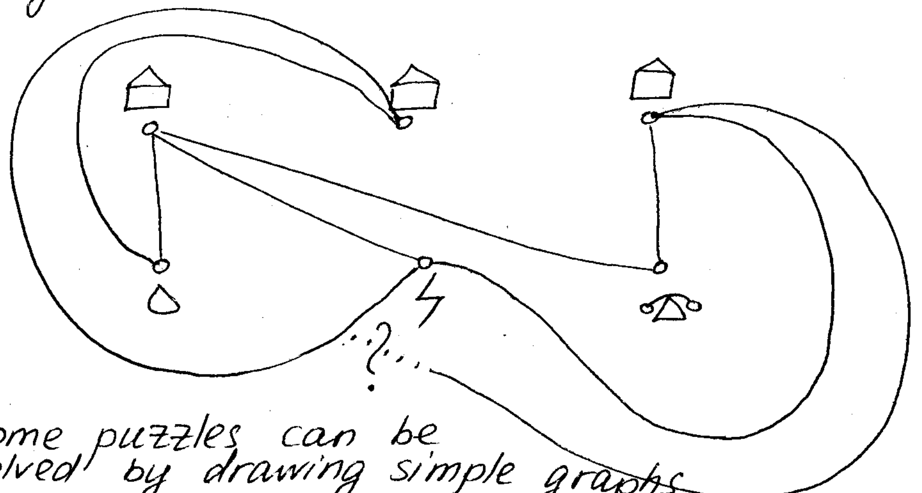
Euler's idea was to realize that land, water and bridges could be modeled by the following graph



The land masses are represented by small circles (or vertices) and the bridges by lines (or edges) which can be straight or curved.

By means of this graph, the physical problem is transformed into this mathematical one: Given the graph above, is it possible to choose a vertex, then to proceed along the edges one after the other and return to the chosen vertex covering every edge exactly once? Euler was able to show that this was not possible.

The Three Houses – Three Utilities Problem is another physical situation which can be modeled by means of a graph. There are three houses, each of which is to be connected to each of three utilities - water, electricity, and telephone - by means of underground pipes. Is it possible to make these connections without any crossovers?



The houses and the utilities are represented by vertices and the pipes are the lines drawn between the vertices.

(The answer to this problem is no).

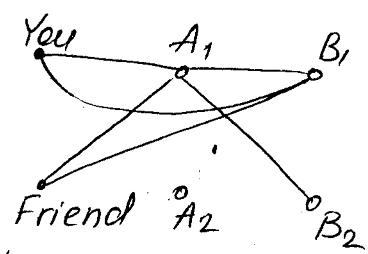
Some puzzles can be solved by drawing simple graphs.

Problem 1. You and your friend return home after a semester at college and are greeted at the airport by your mothers and your friend's two sisters. No doubt, there is a certain amount of hugging! Later, the other five people tell you the number of hugs they got and, curiously, these numbers are all different. Assume that you and your friend did not hug each other, your mothers did not hug each other, and your friend's sisters did not hug each other. Assume also that the same two people hugged at most once. How many people did you hug? How many people did hug your friend?

Solution.

We have six people: you, friend, mothers A_1, A_2, sisters B_1, B_2. Each person did not hug herself and one more person. Therefore the set of answers of number of hugs is $\{0,1,2,3,4\}$.

Next, it is important to realize that the person who hugged four other persons could not be your friend. Otherwise, nobody will report 0 hugs obtained.



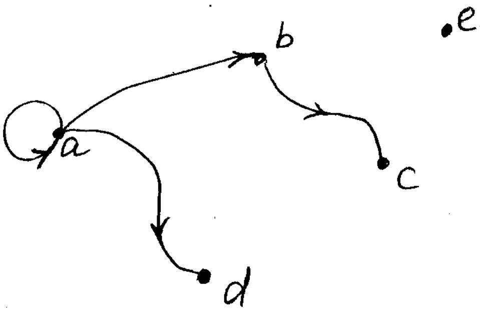
So the person who got four hugs is among the group A_1, A_2, B_1, B_2. Suppose A_1 got four hugs. Since A_1 did not hug A_2, the hugging involving A_1 are uniquely determined

Since one of the group A_1, A_2, B_1, B_2. received no hugs, it is obvious that it must be A_2. Somebody receives three hugs. If it were your friend then nobody will obtain exactly one hug. Thus, B_1 or B_2 obtained three hugs. Let B_1 be a person with three hugs. Now your friend hugged two people, one sister hugges 1 person, another sister hugged three people, one mother hugges 4 people, another did not hugged anyone.
Answers: you and your friend each hugged two people.

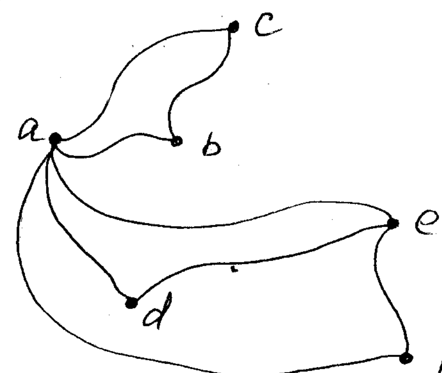
Basic terminology.

Definition. Let V be a finite nonempty set, and let $E \subset V \times V$ be a binary relation on V. The pair $G=(V,E)$ is called a directed graph, where V is the set of vertices and E is its set of edges (or arcs).

When there is no concern about the direction of any edge, we still write $G=(V,E)$. But now E is a set of unordered pairs of elements taken from V. In this case G is called an undirected graph



$V=\{a,b,c,d,e\}$
$E=\{(a,a),(a,b),(a,d),(b,c)\}$.
$G=(V,E)$ is a directed graph

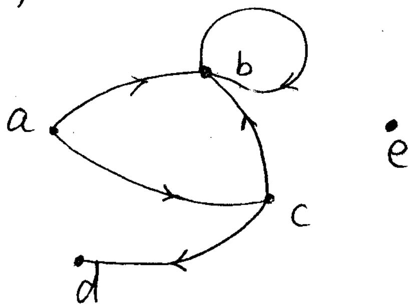


$V=\{a,b,c,d,e,f\}$
$E=\{\{a,b\},\{a,c\},\{a,e\},\{a,d\},\{a,f\},\{b,c\},\{d,e\},\{e,f\}\}$.
$G=(V,E)$ is an undirected graph.

Definition. Let (V,E) be a directed graph and (a,b) be an edge, i.e. $(a,b) \in E$. We say that edge (a,b) is incident with vertices a, b. Sometimes we want to be more specific. We say that the edge (a,b) is incident from a and is insident into b. The vertex a is called the initial vertex

and the vertex b is called the terminal vertex of the edge (a, b).
An edge that is incident from and into the same vertex, like (a,a), is called a loop.
A vertex is said to be an isolated vertex if there is no edge incident with it.

Example.



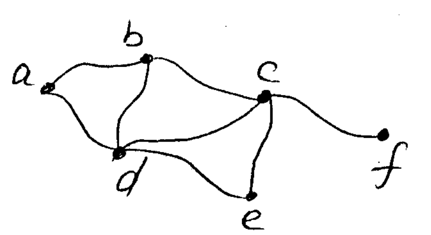
- e is an isolated vertex
- (b,b) is a loop
- for the edge (c, b) c is the initial vertex, b is the terminal vertex.
- there is no an edge incident from d to b.

Definition. Let x, y be (not necessarily distinct) vertices in an undirected graph $G=(V,E)$. An x-y walk in G is a (loop-free) finite sequence

$$x = x_0, e_1, x_1, e_2, x_2, e_3, \dots, e_{n-1}, x_{n-1}, e_n, x_n = y$$

of vertices and edges from G starting at x and ending at y involving the n edges $e_i = \{x_{i-1}, x_i\}$, $1 \le i \le n$.
The length of this walk is n, the number of edges in the walk.
Any x-y walk where $x=y$ is called a closed walk. Otherwise the walk is called open.

Example.



1) $\{a,b\}, \{b,d\}, \{d,c\}, \{c,e\}, \{e,d\}, \{d,b\}$ is an $a-b$ walk of length 6 in which the edge $\{b,d\}$ is met twice.
2) $b \to c \to d \to e \to c \to f$ is an b-f walk of length 5. Here no edge appears more than once. However, vertex c appears twice
3) $\{f,c\}, \{c,e\}, \{e,d\}, \{d,a\}$ is a f-a walk of length 4 with no repetition of either vertices or edges.

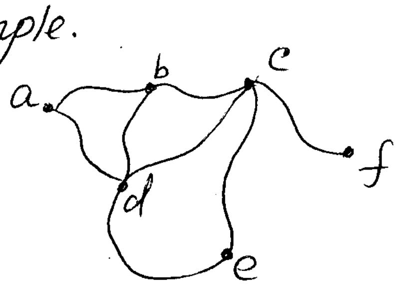
4) $\{b,c\}, \{c,d\}, \{d,b\}$ is a b-b (closed) walk.

Definition. Consider any x-y walk in an undirected graph $G=(V,E)$.

(a) If no edge in the x-y walk is repeated, then the walk is called an x-y trail. A closed x-x trail is called a circuit. [sə:kit] кругооборот

(b) If no vertex of the x-y walk occurs more than once, then the walk is called an x-y path. When $x=y$, the term cycle is used to describe such a closed path.

Example.



1) $b \to c \to d \to e \to c \to f$ is a b-f trail but is not a b-f path because of the repetition of vertex c.

2) $f \to c \to e \to d \to a$ is both an f-a trail and and f-a path.

3) $\{a,b\}, \{b,d\}, \{d,c\}, \{c,e\}, \{e,d\} \{d,a\}$ is an a-a circuit but not an a-a cycle (the vertex d is repeated)

4) $\{a,b\}, \{b,c\}, \{c,d\}, \{d,a\}$ is an a-a cycle.

Remark. For a directed graph, directed walks, directed trails, directed paths, directed circuits, directed cycles are also defined in similar way

Theorem 1. Let $G=(V,E)$ be an undirected graph, with $a, b \in V$, $a \neq b$. If there is a trail (in G) from a to b then there is a path (in G) from a to b.

Proof. Since there is a trail from a to b, we select one of shortest length, say

$$\{a, x_1\}, \{x_1, x_2\}, \ldots\ldots, \{x_n, b\}.$$

If this trail is not a path, we have the situation $\{a, x_1\}, \{x_1, x_2\}, \ldots, \{x_{k-1}, x_k\}, \{x_k, x_{k+1}\}, \ldots, \{x_{m-1}, x_m\}, \{x_m, x_{m+1}\}, \ldots, \{x_n, b\}$, where $k < m$ and $x_k = x_m$. But then we have a contradiction because $\{a, x_1\}, \{x_1, x_2\}, \ldots, \{x_{k-1}, x_k\}, \{x_m, x_{m+1}\}, \ldots, \{x_n, b\}$ is a shorter trail from a to b ∎

Definition. Let $G=(V,E)$ be an undirected graph. We call G connected if there is a path between any two distinct vertices of G. A graph that is not connected is called disconnected.

HW:

Problem 1. Let $G=(V,E)$ be a loop-free connected undirected graph, and let $\{a,b\}$ be an edge of G. Prove that $\{a,b\}$ is a part of a cycle if and only if its removal (the vertices a and b are left) does not disconnect G).

Problem 2. If $G=(V,E)$ is an undirected graph with $|V|=v$, $|E|=e$, and no loops, prove that $2e \le v^2 - v$.

Problem 3. Give an example of a connected graph where removing any edge of G results in a disconnected graph.