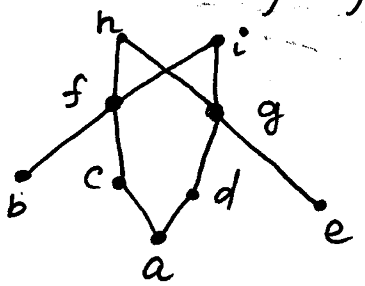
Lattices.

<u>Definition</u>. Let $(A, \leq)$ be a poset. An element $a \in A$ is called a maximal element if for no $b \in A$, $a \neq b$, $a \leq b$. An element a in A is called a minimal element if for no $b \in A$, $a \neq b$, $b \leq a$.



a, b, e are minimal elements, h, i are maximal elements for $(A, \leq)$ with the Hasse diagram as given here.

<u>Definition</u>. Let $(A, \leq)$ be a poset. An element c is called an upper bound of a and b if $a \leq c$ and $b \leq c$. An element c is said to be a least upper bound of a and b if c is an upper bound of a and b, and if there is no other upper bound d of a and b such that $d \leq c$.
Similarly, an element c is called a lower bound of a and b if $c \leq a$ and $c \leq b$, and an element c is called a greatest lower bound of a and b if c is a lower bound of a and b, and if there is no other lower bound d of a and b such that $c \leq d$.

Example. Let $(A, \leq)$ be a poset with the Hasse diagram as above. Then

h is a least upper bound of f and g,
i is another least upper bound of f and g,
f is a greatest lower bound of h and i,
g is another greatest lower bound of h and i.

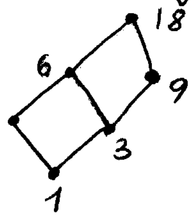
<u>Definition</u>. A partially ordered set is said to be a lattice if every two elements in the set have a unique least upper bound and a unique greatest lower bound.

Examples ① $A_1 = \{1, 2, 3, 6, 9, 18\}$,

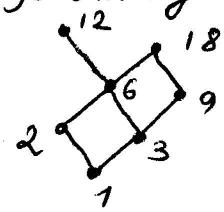
$R = \{(x, y): x, y \in A_1, x \text{ divides } y\}$.

We know that (A_1, R) is a poset.

For every pair $a \in A$ and $b \in A$, 1 is a lower bound and 18 is an upper bound. Clearly, every two elements in A have a unique least upper bound and a unique greatest lower bound. Therefore (A_1, R) is a lattice.



② $A_2 = \{1, 2, 3, 6, 9, 12, 18\}$, $R = \{(x, y): x, y \in A, x \text{ divides } y\}$. (A_2, R) is a poset. The Hasse diagram for (A_2, R) is the following



There is no upper bound for 12 and 18, for 12 and 9. Therefore, (A_2, R) is not a lattice.

<u>Problem 1.</u> Let (A, R_1) be a poset. Let

$R_2 = \{(b, a): b, a \in A, (a, b) \in R_1\}$ be a binary relation on A

(a) Show that (A, R_2) is a poset.

(b) Show that if (A, R_1) is a lattice then (A, R_2) is a lattice

Solution:

(a) - R_1 is a partial order $\Rightarrow \forall a \in A, (a, a) \in R_1 \Rightarrow$ $\forall a \in A, (a, a) \in R_2 \Rightarrow R_2$ is reflexive.

- Let $(a, b) \in R_2$ and $(b, a) \in R_2$. Then $(b, a) \in R_1$ and $(a, b) \in R_1$. Since R_1 is a partial order, then it is antisymmetric, and so, $a = b$. Hence, R_2 is antisymmetric.
- Let $(a, b) \in R_2$, $(b, c) \in R_2 \Rightarrow (b, a) \in R_1$ and $(c, b) \in R_1$. By transitivity of R_1, $(c, a) \in R_1 \Rightarrow (a, c) \in R_2$. Then R_2 is transitive.

Totally, (A, R_2) is a poset.

(b) Let a and b be two arbitrary elements of A. Since (A, R_1) is a lattice, then $\exists$ a unique least upper bound c and a unique greatest lower bound d of a and b (with respect to R_1), i.e.

$$a \underset{R_1}{\leq} c, \quad b \underset{R_1}{\leq} c, \quad \nexists\ c_1 \underset{R_1}{<} c \text{ s.t. } a \underset{R_1}{\leq} c_1, \quad b \underset{R_1}{\leq} c_1$$

$$a \underset{R_1}{\geq} d, \quad b \underset{R_1}{\geq} d, \quad \nexists\ d_1 \underset{R_1}{>} d \text{ s.t. } a \underset{R_1}{\geq} d_1, \quad b \underset{R_1}{\geq} d_1$$

We can rewrite these relations in terms of R_2 in the following form

$$(*) \quad a \underset{R_2}{\geq} c, \quad b \underset{R_2}{\geq} c, \quad \nexists\ c_1 \underset{R_2}{>} c \text{ s.t. } a \underset{R_2}{\geq} c_1, \quad b \underset{R_2}{\geq} c_1$$

$$(**) \quad a \underset{R_2}{\leq} d, \quad b \underset{R_2}{\leq} d, \quad \nexists\ d_1 \underset{R_2}{<} d \text{ s.t. } a \underset{R_2}{\leq} d_1, \quad b \underset{R_1}{\leq} d_1$$

It follows from $(*)$ that a, b have a unique greatest lower bound, namelly c, with respect to R_2.
It follows from $(**)$ that a, b have a unique least upper bound d with respect to R_2.
Since a and b were arbitrary elements of A then (A, R_2) is a lattice.

Problem 2. Let (A,R) be a poset. Prove or disprove each of the following statements.
(a) If (A, R) is a lattice, then it is a total order
(b) If (A, R) is a total order, then it is a lattice.

Solution.
(a) is not true, see Example ①.
$A = \{1,2,3,6,9,18\}$, $R = \{(x,y) : x \text{ divides } y\}$.
(A, R) is a lattice but not a totally ordered set.
(b) Let (A,R) be a totally ordered set. Then for any two elements a, b we have
(i) either $a < b$, or (ii) $b < a$
In case (i), gr. lower bound is a, least upper bound is b.
In case (ii), — // — b, — // — a.

Functions.

Function $f: A \to B$ is often defined as a rule which associates to each element of a set A an element of another set B. The essential characteristic of a function is that the value which it associates with a given element is uniquely determined by that element.

Definition. A function from a set A to a set B is a binary relation R from A to B with the property that, $\forall a \in A$, there is exactly one $b \in B$ s.t. $(a, b) \in R$.

For a function R from A to B, we also use notation $R(a) = b$, where element b is called the image of a under R.

Definition. The set A is called the domain of the function R and the set B is called the range of the function R.

Example. ① $A = \{1, 2, 3\}$, $B = \{x, y\}$,

$$R_1 = \{(1, x), (2, y), (3, x)\}.$$

R_1 is a function from A to B, $R(1) = x$, $R(2) = y$, $R(3) = x$.

② $R_2 = \{(1, x), (3, y)\}$.

R_2 is not a function from A to B because R_2 contains no ordered pair with first coordinate 2.

③ $R_3 = \{(1, x), (2, x), (3, y), (2, y)\}$.

R_3 is not a function from A to B because 2 is the first coordinate of two ordered pairs.

Problem 1. Suppose A is the set of surnames of people listed in the Ankara telephone directory. Is it likely that

$$f = \{(a, n) : a \text{ is on page } n\}$$

is a function from A to the set of natural numbers?

Solution: It is almost certain that some surnames are listed on a number of different pages. Then it is unlikely that f is a function.

Sometimes (often in calculus), a function is sufficiently nice that is possible to write down a precise formula showing how $R(x)$ is determined by x, for example,

$R_1(x) = x^3$, $\quad R_2(x) = 3x-7$, $\quad R_3(x) = \ln x$.

We are really talking about binary relations

$R_1 = \{(x, x^3) : x \in \mathbb{R}\}$, $R_2 = \{(x, 3x-7) : x \in \mathbb{R}\}$, $R_3 = \{(x, \ln x) : x > 0\}$.

When these functions are graphed as usual in xy-plane, we are in actual fact, making a picture of the ordered pairs $(x, R_i(x))$, $i = 1,2,3$, i.e. picture binary relations R_i, $i = 1, 2, 3$.

Definition. A function from A to B is said to be an onto function if every element of B is the image of one or more elements of A.

A function f from A to B is said to be a one-to-one (1-1) function if no two elements of A have the same image, i.e.

$$a_1 \neq a_2 \implies f(a_1) \neq f(a_2),$$

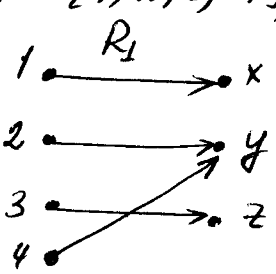
equivalently (taking the contrapositive)

$$f(a_1) = f(a_2) \implies a_1 = a_2.$$

A function from A to B is said to be a one-to-one onto function if it is both an onto and a one-to-one function.

Examples.

① $A = \{1, 2, 3, 4\}$, $\quad B = \{x, y, z\}$



$R_1 = \{(1,x), (2,y), (3,z), (4,y)\}$

R_1 is a function

R_1 is onto but not one-to-one.

Remark: For given A and B, there is no one-to-one function from A to B.

② $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 2x - 3$.

Note that

$\forall a \in \mathbb{Z}$, $2a-3$ is an odd number.

Therefore, f is not onto (since for all even numbers b there are no $a \in \mathbb{Z}$ s.t. $f(a)=b$). Function f is one-to-one, however:

$$f(a_1)=f(a_2) \Leftrightarrow 2a_1-3=2a_2-3 \Longrightarrow a_1=a_2.$$

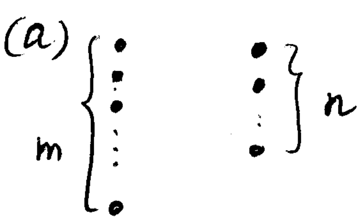
③ $f: \mathbb{N} \longrightarrow \mathbb{N}$, $f(x)=2x-3$.

Note that $f(1)=-1$, and $-1 \notin \mathbb{N}$. Hence, f is not a function.

<u>Problem 2(a)</u> Given sets X and Y with $|X|=m$ and $|Y|=n$ guess a general formula for the number of functions from X to Y (b) How many one-to-one functions from X to Y do we have?

<u>Solution:</u>

(a)



$X=\{a_1,a_2,\dots\ a_m\}$

Let f be arbitrary function from X to Y.

$f(a_1)$ can be any of n values of Y

$f(a_2)$ —— // ——

⋮

$f(a_m)$ —— // ——

Totally, there are n^m different functions $X \to Y$.

(b) Let f be a one-to-one function. Then

$f(a_1)$ can be one of n elements of Y,

$f(a_2)$ can be chosen in $(n-1)$ ways (since one element of Y is already image of a_1)

⋮

—— // ——

⋮

$f(a_m)$ is one of $(n-(m-1))$ elements of Y.

Totally, there are $n\cdot(n-1)\cdot\ldots\cdot(n-m+1)$ different one-to-one functions from X to Y.

Note: 1) If $m=n$, the number of one-to-one functions is $n!$

2) If $m<n$, the number of one-to-one functions is 0.

Problem 3. Suppose that A and B are sets with the same finite number of elements and that $f: A \to B$ is a function. Prove that f is one-to-one if and only if f is onto.

Proof.

(i) We have $|A| = |B| = n$. Let f be one-to-one. We will show that f is onto.
Assume that f is not onto. Then $\exists\, b \in B$ s.t. $f(a) \neq b$, $\forall a \in A$. Hence, $f: A \to B \setminus \{b\}$. It can not be one-to-one since $|A| = n$ but $|B \setminus \{b\}| = n-1 < n$. This contradiction implies that our assumption was wrong. Therefore, f is onto.

(ii) Let f be onto. Let us show that f is one-to-one. Assume that it is not one-to-one. Then $\exists\, a_1, a_2 \in A$ s.t. $f(a_1) = f(a_2)$. Then $|f(A)| \leq n-1$, where $f(A) = \{b \in B : b = f(a), a \in A\}$. Therefore, f is not onto. It implies that f is one-to-one. □

The Pigeonhole Principle.

Suppose that B is a set of bird houses, A is a set of pigeons, and f is a function which assigns to each pigeon a bird house. Pigeon-Hole Principle says that if $|A| > |B|$, i.e. we have more pigeons than the bird houses, then at least two pigeons will be assigned to the same house.

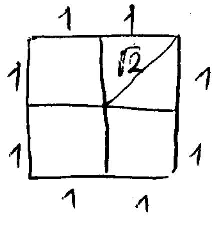
The pigeonhole principle. If $|A| > |B|$, then for any function f from A to B, there exist $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$.

In other words, if $|A| > |B|$, function $f: A \to B$ can not be one-to-one.

Example 1. Application of the principle:
Among 13 people, there are at least two of them who were born in the same month.

Example 2. Given five points inside a square whose sides have length 2, prove that two are within $\sqrt{2}$ of each other.

Solution. Subdivide the square into four squares with sides of length 1. By the Pigeonhole Principle, at least two of the five chosen points must lie in, or on boundary of, the same smaller square. But these points are at most $\sqrt{2}$ apart (the length of the diagonal of a smaller square.



Problem 1. Prove that in any list of ten natural numbers, $a_1, a_2, \dots, a_{10}$, there is a string of consecutive items of the list, $a_\ell, a_{\ell+1}, a_{\ell+2}$"; whose sum is divisible by 10. (We include the possibility that the "string" consists of just one number).

Solution. Consider the ten numbers

$$a_1, \quad a_1+a_2, \quad \dots, \quad a_1+a_2+\dots+a_{10}.$$

If any of these is divisible by 10, we have the desired conclusion;

Otherwise, there are at least two numbers, say

$a_1+a_2+\dots+a_s$ and $a_1+a_2+\dots+a_t$, $s<t$, that have the same remainder after division by 10, i.e.

$$a_1+a_2+\dots+a_s = 10k_1+r \quad \text{and} \quad a_1+a_2+\dots+a_t = 10k_2+r.$$

Then their difference

$$-(a_1+a_2+\dots+a_s)+(a_1+a_2+\dots+a_t) = a_{s+1}+a_{s+2}+\dots+a_t = 10(k_2-k_1)$$

is divisible by 10. Again we reach the desired conclusion.

Problem 2. Marina has three weeks to prepare for a tennis tournament. She decides to play at least one set every day but not more than 36 sets in all. Show that there is a period of consecutive days during which she will play exactly 5 sets.

Solution. Let a_i denote the total number of sets she plays up through the ith day. Clearly, $a_1, a_2, \dots, a_{21}$ is strictly increasing sequence, with $a_1 \geq 1$ and $a_{21} \leq 36$. Consider another strictly increasing sequence $a_1+5, a_2+5, \dots, a_{21}+5$. We have $a_{21}+5 \leq 41$. Since the values of 42 numbers $a_1, a_2, \dots, a_{21}, a_1+5, a_2+5, \dots, a_{21}+5$ range from 1 to 41, then two of them must be the same. Moreover, because both the sequences $a_1, a_2, \dots, a_{21}$ and $a_1+5, a_2+5, \dots, a_{21}+5$ are increasing, we have that

$$a_i = a_j + 5 \quad \text{for some } a_i \text{ and } a_j,$$

i.e.

$$a_i - a_j = 5,$$

From the $(j+1)$th day to (i)th day Marina will play exactly 5 sets.

Problem 3. Show that in a sequence of n^2+1 distinct integers, there is either an increasing subsequence of length $(n+1)$ or a decreasing subsequence of length $n+1$.

Solution. Let $a_1, a_2, \dots, a_{n^2+1}$ denote the sequence of integers. Let us associate an ordered pair (x_k, y_k) with each integer a_k, where x_k is the length of a longest decreasing subsequence starting at a_k, and y_k is the length of a longest increasing subsequence starting at a_k.

Suppose there is no increasing subsequence or decreasing subsequence of length $n+1$ in the sequence $a_1, \dots, a_{n^2+1}$. Then $1 \leq x_k \leq n$ and $1 \leq y_k \leq n$, $\forall k = 1, 2, \dots, n^2+1$. We have therefore n^2+1 ordered pairs (x_k, y_k). But only n^2 of them can be distinct. Then there must exist a_i and a_j with the same ordered pairs $(x_i, y_i) = (x_j, y_j)$, $i<j$. However, this is impossible, because if $a_i < a_j \Rightarrow x_i > x_j$ and if $a_i > a_j \Rightarrow y_i > y_j$. Therefore our assumption that there is no incr. or decreasing subsequences of length $n+1$ was wrong.