Partial Ordering Relations

Definition. A binary relation R on a set A is said to be an antisymmetric relation if $(a,b)\in R$ implies that (b,a) is not in R unless $a=b$.

Examples.

1) $A=\mathbb{R}$, $R=\{(x,y): x\le y\}$ is an antisymmetric relation on $\mathbb{R}$ since $x\le y$ and $y\le x$ implies $x=y$.

2) $A=\mathcal{P}(S)$ is a set of all subsets of S,
$R=\{(X,Y): X,Y\in\mathcal{P}(S),\ X\subset Y\}$ is an antisymmetric relation on $\mathcal{P}(S)$ since $X\subset Y$ and $Y\subset X$ implies $X=Y$.

Definition. A binary relation R on a set A is said to be a partial ordering relation if it is reflexive, antisymmetric and transitive.
A partially ordered set, poset for short, is a pair (A,R), where R is a partial ordering relation on A.

Examples.

1) $A=\mathbb{R}$, $R=\{(x,y): x^2\le y^2\}$.
R is reflexive, transitive, but not antisymmetric.
Indeed, $(-4,4)\in R$, $(4,-4)\in R$ but $4\neq -4$.
So, (A,R) is not a poset.

2) $A=\mathbb{Z}\times\mathbb{Z}$, $R=\{(a,b): a\in A,\ b\in A,\ a=(a_1,a_2),\ b=(b_1,b_2),$
$a_1\le b_1,\ a_1+a_2\le b_1+b_2\}$

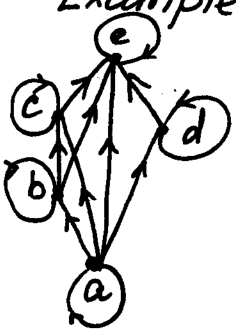
R is reflexive since $(a,a)\in R$ $\forall a=(a_1,a_2)\in\mathbb{Z}\times\mathbb{Z}$
R is transitive. Indeed, let $(a,b)\in A$, $(b,c)\in A$, $a=(a_1,a_2)$, $b=(b_1,b_2)$, $c=(c_1,c_2)$. Then $a_1\le b_1\le c_1$, i.e. $a_1\le c_1$; $a_1+a_2\le b_1+b_2\le c_1+c_2$, i.e. $a_1+a_2\le c_1+c_2$.
It implies that $(a,c)\in A$ whenever $(a,b)\in A$, $(b,c)\in A$

Let us show that R is antisymmetric.
Assume that $(a,b)\in R$ and $(b,a)\in R$, $a=(a_1,a_2)$, $b=(b_1,b_2)$. Then $a_1\le b_1$ and $b_1\le a_1$, i.e. $a_1=b_1$.
Also, $a_1+a_2\le b_1+b_2$ and $b_1+b_2\le a_1+a_2$, i.e. $b_1+b_2=a_1+a_2$. The last equality together with $a_1=b_1$ implies $a_2=b_2$. Therefore $(a_1,a_2)=(b_1,b_2)$, i.e. $a=b$.
It proves that R is antisymmetric.
Thus, (A,R) is a poset.

<u>Notation.</u> Let (A,R) be a poset. For each ordered pair (a,b) in R, we write $a\le b$ instead of $(a,b)\in R$. We also write $(A,\le)$ instead of (A,R).

<u>Definition.</u> Let $(A,\le)$ be a poset. A subset of A is called a chain if every two elements in the subset are related.

Example.



$A=\{a,b,c,d,e\}$
R is a partial ordering relation on A, since it is reflexive, antisymmetric and transitive.
$S_1=\{a,b,c,e\}$, $S_2=\{b,c,e\}$, $S_3=\{a,d,e\}$ are chains.
$S_4=\{a,b,d\}$ is not a chain.

<u>Definition.</u> A poset $(A,\le)$ is called a totally ordered set if set A is a chain. In this case, the binary relation $\le$ is called a total ordering relation.

Examples.

1. The real numbers are totally ordered by $\le$ because for every pair a,b of real numbers either $a\le b$ or $b\le a$.
2. The set of sets $\{\{a\},\{b\},\{c\},\{a,c\}\}$ is not totally ordered by $\subset$ since neither $\{a\}\subset\{b\}$ nor $\{b\}\subset\{a\}$.

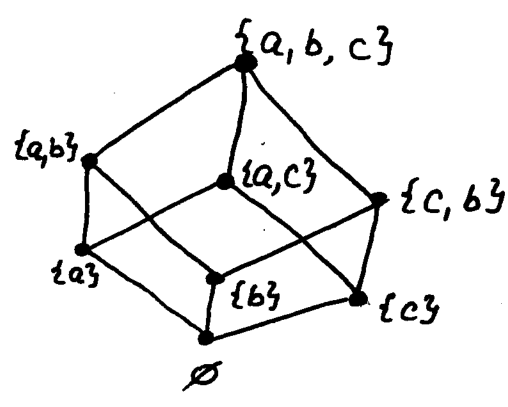
Partial orders are often pictured by means of a diagram named after Helmut Hasse (1898-1979). In the Hasse diagram of a poset $(A, \leq)$,

- there is a dot (or vertex) associated with each element of A;
- if $a \leq b$, then the dot for b positioned higher than the dot for a; and
- if $a \leq b$, $a \neq b$ and there is no intermediate c with $a < c < b$ then a line is drawn from a to b.

Examples.

1. $A = \{0, 1, 2, 3\}$.
The Hasse diagram for $(A, \leq)$

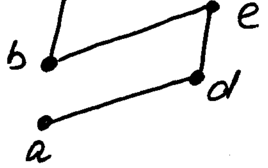
2. $A = \mathcal{P}(\{a, b, c\})$
The Hasse diagram for $(\mathcal{P}(\{a, b, c\}), \subset)$.



Remark. Knowing only the Hasse diagram for a poset (A, R) we can write all the elements of R.

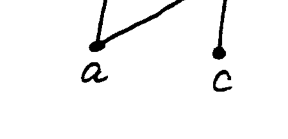
Problem. List all pairs (x, y) with $x \leq y$ in the partial order described by the following Hasse diagrams

1)



$R = \{(a,a), (b,b), (c,c), (d,d), (e,e), (f,f), (a,d), (d,e), (a,e), (b,e), (b,c), (c,f), (b,f)\}$.

2)



$R = \{(a,a), (b,b), (c,c), (d,d), (a,b), (a,d), (c,d)\}$.

Binary Relations. Review.

Definition. Let (A, R) be a poset. If A is a chain then (A, R) is called a totally ordered set.

A is totally ordered $\Longleftrightarrow$ $\forall x \in A$, $\forall y \in A$, we have that $(x,y) \in R$, or $(y,x) \in R$ or both.

Problem 1.

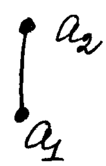
(a) Described the structure of the Hasse diagram for a totally ordered poset (A, R), where $|A| = n$

(b) For a set A where $|A| = n \geq 1$, how many relations on A are total orders?

Solution.

(a) For $n = 1$, $A = \{a\}$ and the Hasse diagram is 

For $n=2$, $A = \{a_1, a_2\}$, $R = \{(a_1,a_1), (a_2,a_2), (a_1,a_2)\}$ or $R = \{(a_1,a_1), (a_2,a_2), (a_2,a_1)\}$.

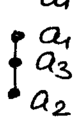
The Hasse diagram is either 

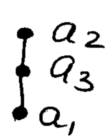
or 

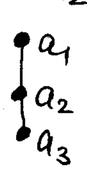
For $n = 3$, $A = \{a_1, a_2, a_3\}$. We have the following possibilities

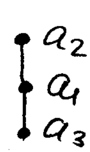
$a_1 < a_2 < a_3 \longleftrightarrow$ 

$a_2 < a_1 < a_3 \longleftrightarrow$ 

$a_2 < a_3 < a_1 \longleftrightarrow$ 

$a_1 < a_3 < a_2 \longleftrightarrow$ 

$a_3 < a_2 < a_1 \longleftrightarrow$ 

$a_3 < a_1 < a_2 \longleftrightarrow$ 

For $n=1,2,3$, we see that in the Hasse diagram n elements are arranged along a vertical line. Let us prove that the same holds for any $n \geqslant 1$. We will use the Principle of Mathematical Induction.

We already checked that the base hold for $n=2$. Assume that the statement is true for some n, $n \geqslant 1$. Let us show that the statement is true for $n+1$.

Let $|A| = n+1$, i.e. $A = \{a_1, a_2, \ldots, a_{n+1}\}$

We have that poset (A, R) is totally ordered. Take element a_1 and compare it with the other elements. If $a_1 > a_i$, $\forall\ i=2,3,\ldots,n+1$, then define b to be a_1. If not, take element a_2 and compare it with the other elements. If $a_2 > a_i$, $\forall\ i=1,3,4,\ldots,n+1$, then define b to be a_2. If not, take element a_3, and follow the procedure until we find the element a_j s.t. $a_j > a_i$, $\forall\ i=1,2,\ldots,n+1$ but $i \neq j$. Define $b = a_j$.

Such defined b is one of elements of A and $(a_i, b) \in R$ $\forall\ i=1,2,\ldots,n+1$.

We have that $|A \setminus \{b\}| = n$ and by our assumption the Hasse diagram for $A \setminus \{b\}$ is a vertical line joining n elements of $A \setminus \{b\}$.

To get the Hasse diagram of A we continue this vertical line and join it with element b. It finishes the proof.

(b) By part (a), the number of totally ordered sets on A, $|A| = n$, is the number of permutations of n elements. It is $n!$

Problem 2. Let $A = \{1, 2, 4, 6, 8\}$ and for $a, b \in A$ define $a \leq b$ if and only if $\frac{b}{a}$ is an integer.

(a) Prove that $\leq$ defines a partial order on A

(b) Draw the Hasse diagram for $\leq$.

(c) Is $(A, \leq)$ totally ordered. Explain.

Solution:

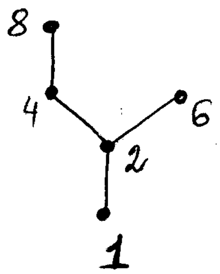
(a) R is reflexive since $\forall a \in A$, $(a,a) \in R$, i.e. $a \leq a$ (because $\frac{a}{a} = 1$)

R is transitive since if $a \leq b$, $b \leq c$, then $\frac{b}{a} \in \mathbb{Z}$, $\frac{c}{b} \in \mathbb{Z}$ and therefore $\frac{c}{a} = \frac{b}{a} \cdot \frac{c}{b} \in \mathbb{Z}$, i.e. $a \leq c$.

R is antisymmetric. Indeed, if $a \leq b$ and $b \leq a$, then $\frac{a}{b}$ and $\frac{b}{a}$ are both integers. It is possible only if $a = b$ or $a = -b$. But a and b are both positive then $a = b$.

Therefore, $\leq$ defines a partial order on A.

(b)



(c) $(A, \leq)$ is not totally ordered since its Hasse diagram is not a vertical line.

Problem 3. Let $|A| = n$, $\mathcal{P}(A)$ be the set of subsets of A, $R = \{(X, Y) ; X, Y \in \mathcal{P}(A); X \subset Y\}$. What is the length of the longest chain in the poset $(\mathcal{P}(A), R)$.

Solution. Let $A = \{a_1, a_2, \dots, a_n\}$, $|\mathcal{P}(A)| = 2^n$

Let S be an arbitrary chain in $\{\mathcal{P}(A) R)\}$. Then it is a subset of $\mathcal{P}(A)$. Its length $\leq 2^n$.

For example, for $n = 2$, $A = \{a, b\}$, $\mathcal{P}(A) = \{\{a\}, \{b\}, \emptyset, \{a, b\}\}$.

$S_1 = \{\{a\}, \{a, b\}, \emptyset\}$ and $\{\emptyset, \{b\}, \{a, b\}\}$ are two chains with the largest length, that is 3 (we have $3 \le 2^2$).

Any chain in $(\mathcal{P}(A), R)$ is of the form

$$A_{i_1} < A_{i_2} < \dots < A_{i_m}, \text{ where } A_{i_j} \in \mathcal{P}(A).$$

Obviously,

$$0 \le |A_{i_1}| < |A_{i_2}| < \dots < |A_{i_m}| \le n$$

Any chain S with the largest length satisfies $|A_{i_1}| = 0$, i.e. $A_{i_1} = \emptyset$,

$|A_{i_2}| = 1$, $|A_{i_3}| = 2$, ..., $|A_{i_m}| = n \Rightarrow m = n+1$.

Therefore, the length of the longest chain is $n+1$.

Problem 4. Let A be the set of points different from the origin in the Euclidean plane. For $p, q \in A$, define $(p, q) \in R$ if $p = q$ or the line through the distinct points p and q passes through the origin.

(a) Prove that R is an equivalence relation on A

(b) Find the equivalence classes of R.

Solution.

(a) - By the definition of R, $(p,p) \in R$, $\forall p \in A$. Therefore R is reflexive.

- Let $(p,q) \in R$. Then either $p = q$, or $p, q, 0$ lie on the same line. Therefore, $(q, p) \in R$. Hence R is symmetric

- Let $(p,q) \in R$ and $(q,r) \in R$. Then there are the following possibilities

(i) $p = q$ $\longrightarrow$ (i 1) $p = q = r$

$\longrightarrow$ (i 2) $p = q \neq r$, $\exists$ line through $0, \overset{p}{\overset{\shortparallel}{q}}, r$

(ii) $p \neq q$, $\exists$ line through $0, p, q$ $\longrightarrow$ (ii 1) $p \neq q$, $\exists$ line through $0, p, \underset{\overset{\shortparallel}{r}}{q}$, $q = r$

$\longrightarrow$ (ii) $p \neq q$, $\exists$ line through $0, p, q$, $q \neq r$, $\exists$ line through $0, q, r$.

In case (i1), $(p, r) \in R$
In case (i2), $(p, r) \in R$
In case (i3), $p \neq r$ and $(p, r) \in R$
In case (i4), either $p = r$ and then $(p, r) \in R$,
or $p \neq r$ and $\exists$ a unique line through O and p and it contains q.
This line can be described as a line through O and q. We know it contains r as well.
Therefore $\exists$ line through O, p, r. Then $(p, r) \in R$.
Hence, R is transitive.
Totally, R is an equivalence relation on A.

(b). Let p be any element of A.

$[(p_1, p_2)] = [p] = \{ q \in A : p = q$, or p, q, O lie on the same line$\}$.
$= \{(x, \frac{p_2}{p_1} x) : x \in \mathbb{R}, x \neq 0\}$ = the line passing through O and p, with deleted point O.

Equivalence classes of R are lines passing through the origin with one removed point $(0,0)$ from them.

<u>Problem 4.</u> Let $A = \mathbb{Z}$, $R = \{(a, b) ; 3a + b$ is a multiple of $4\}$.
Prove that R is an equiv. relation. Find $[0]$, $[2]$.

<u>Solution.</u>

- $\forall a \in \mathbb{Z}$, $(a, a) \in R$. Indeed, $3a + a = 4a$ is a multiple of 4 $\forall a \in \mathbb{Z}$. Therefore, R is reflexive.
- Let $(a, b) \in R \Rightarrow 3a + b = 4k$ for some $k \in \mathbb{Z}$. Then $3b + a = 3(b + 3a) - 8a$ is divisible by 4. So, $(b, a) \in R \Rightarrow R$ is symmetric.
- Let $(a, b) \in R$, $(b, c) \in R$. Then $3a + b = 4k$, $3b + c = 4l$, for some $k, l \in \mathbb{Z} \Rightarrow 3a + c = 3a + b + 3b + c - 4c$ $= 4(k + l) - 4c$ is divisible by 4, i.e. $(a, c) \in R$. Then R is transitive. Totally, R is an equiv. relation.

$[0] = \{b : b \text{ is divisible by } 4\} = 4\mathbb{Z}$

$[2] = \{b : 3\cdot 2 + b \text{ is divisible by } 4\} =$

$= \{b : 6+b = 4k,\ k\in\mathbb{Z}\} = \{b : -2+b = 4k,\ k\in\mathbb{Z}\} =$

$= \{4k+2,\ k\in\mathbb{Z}\} = 2+4\mathbb{Z}.$

Problem 6. Let A and R be as in the previous example. Find the partition corresponding to R.

Solution.

The partition of A induced by R is given by equivalence classes of R. We already know classes [0] and [2]. Let us find [1] and [3].

$[1] = \{b : 3+b = 4k,\ k\in\mathbb{Z}\} = \{b : -1+b = 4k,\ k\in\mathbb{Z}\} = 1+4\mathbb{Z}$

$[3] = \{b : 9+b = 4k,\ k\in\mathbb{Z}\} = \{b : -3+b = 4k,\ k\in\mathbb{Z}\} = 3+4\mathbb{Z}.$

We have that $A = \mathbb{Z} = [0]\cup[1]\cup[2]\cup[3]$. Then these equivalence classes give us a partition of $A = \mathbb{Z} = \{4\mathbb{Z}, 1+4\mathbb{Z}, 2+4\mathbb{Z}, 3+4\mathbb{Z}\}$.

Problem 7. List all the equivalence relations on the following sets

(a) $\{a, b\}$, (b) $\{a, b, c\}$.

Solution:

(a) We have 2 different partitions of $\{a, b\} = A$

$A = \{a, b\} = \{a\}\cup\{b\}$

For the partition $\{a, b\}$, the corresponding equiv. relation is $\{(a,a), (a,b), (b,a), (b,b)\}$.

For the partition $\{\{a\}, \{b\}\}$, the corresponding equiv. relation is $\{(a,a), (b,b)\}$.

(b) We have

$\pi_1 = \{\{a\}, \{b\}, \{c\}\}$, $\pi_2 = \{\{a, b\}, \{c\}\}$, $\pi_3 = \{\{a, c\}, \{b\}\}$, $\pi_4 = \{\{c, b\}, \{a\}\}$, $\pi_5 = \{\{a, b, c\}\}$. Each partition produces its own equiv. relation.

Problem 6. For a set A, let $C = \{ \pi : \pi \text{ is a partition of } A\}$

Let R be a binary relation on C s.t.
$(\pi_1, \pi_2) \in R$ iff π_1 is a refinement of π_2, or the same $\pi_1 \leq \pi_2$.

(a) Show that R is a partial order on C.

(b) For $A = \{1,2,3,4,5\}$, let

$\pi_1 = \{\{1,2\}, \{3,4,5\}\}$, $\pi_2 = \{\{1,2\}, \{3,4\}, \{5\}\}$

$\pi_3 = \{\{1\}, \{2\}, \{3,4,5\}\}$; $\pi_4 = \{\{1,2\}, \{3\}, \{4\}, \{5\}\}$.

Draw the Hasse diagram for $(C = \{\pi_1, \pi_2, \pi_3, \pi_4\}, R)$.

Solution:

Remainding.

Let π_1 and π_2 be two partitions of A. Then $\exists$ equivalence relations R_1 and R_2 on A associated with π_1 and π_2. We say $\pi_1 \leq \pi_2$, if $R_1 \subset R_2$.

(a) - Since $R_1 \subset R_1 \Rightarrow \pi_1 \leq \pi_1$ $\forall$ partition π_1 of $A \Rightarrow$ $(\pi_1, \pi_1) \in R$. Therefore R is reflexive.

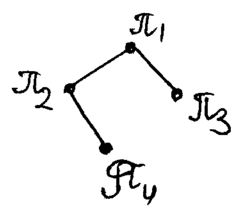
- Let $(\pi_1, \pi_2) \in R$, $(\pi_2, \pi_3) \in R$. Then $R_1 \subset R_2 \subset R_3 \Rightarrow (\pi_1, \pi_3) \in R \Rightarrow R$ is transitive.

- Let $(\pi_1, \pi_2) \in R$ and $(\pi_2, \pi_1) \in R$. Then $R_1 \subset R_2 \subset R_1 \Rightarrow R_1 = R_2 \Rightarrow \pi_1 = \pi_2 \Rightarrow R$ is antisym.

Totally, R is a partial ordering relation on A.

(b) We have

$\pi_2 \leq \pi_1$, $\pi_3 \leq \pi_1$, $\pi_4 \leq \pi_1$, $\pi_4 \leq \pi_2$



HW

Problem 7. Let $A = \{1,2,3,4,5,6,7\}$. For each of the following values of r, determine an equivalence relation R on A with $|R| = r$, or explain why no such relation exists.

(a) $r = 6$, (b) $r = 7$, (c) $r = 8$, (d) $r = 9$, (e) $r = 11$, (f) $r = 23$.