Remainding: An equivalence relation R on a set A is a relation that is reflexive, symmetric and transitive.

Equivalence relations and Partitions

Definition. For a given set A, let I be an index set, $A_i \subset A$ for all $i \in I$ and $A_i \neq \emptyset$ for all $i \in I$. Then $\{A_i\}_{i\in I}$ is called a partition of A if

(a) $A = \bigcup_{i\in I} A_i$;

(b) $A_i \cap A_j = \emptyset$ for all $i, j \in I$ where $i \neq j$.

Example. $A = \{1, 2, 3, 4, 5\}$

1) $A_1 = \{1, 2\}$, $A_2 = \{3, 4, 5\}$ is a partition of A.
2) $A_1 = \{3\}$, $A_2 = \{2, 4\}$, $A_3 = \{1, 5\}$ is a partition of A.

Definition. Each subset A_i in a partition $\{A_i\}_{i\in I}$ of A is called a block of the partition.

Question: How do partitions come into play with equivalence relations?

Definition. Let R be an equivalence relation on a set A. For each $x \in A$, the equivalence class of x, denoted $[x]$, is defined by
$$[x] = \{y \in A : (x, y) \in R\}.$$

Example. $A = \mathbb{Z}$, $R = \{(x, y) : 3 \text{ divides } (x - y)\}$.
For this equivalence relation

$[0] = \{3k : k \in \mathbb{Z}\} = 3\mathbb{Z}$,
$[1] = \{3k+1 : k \in \mathbb{Z}\}$,
$[2] = \{3k+2 : k \in \mathbb{Z}\}$,

$[3] = \{y : 3 - y = 3m; m \in \mathbb{Z}\} = \{3(m+1) : m \in \mathbb{Z}\} = 3\mathbb{Z}$.

Similarly we can show that
$[4]=[1]$, $[5]=[2]$, ... ~~Most important,~~
$\{[0],[1],[2]\}$ provides a partition of $\mathbb{Z}$.

Theorem. Let R be an equivalence relation on a set A, and $x, y \in A$. Then
(a) $x \in [x]$;
(b) $(x,y) \in R$ if and only if $[x]=[y]$;
(c) $[x]=[y]$, or $[x] \cap [y] = \emptyset$.

Proof.
(a) Since R is reflexive then $(x,x) \in R$, $\forall x \in A$ and hence $x \in [x]$.
(b) – Let $(x,y) \in R$. Then, for any $t \in [x]$ we have (x,t) and (t,x) belong to R. By transitivity of R, $(t,y) \in R$ and by symmetry of R, $(y,t) \in R$, i.e. $t \in [y]$. We have that any element of $[x]$ belongs to $[y]$, i.e. $[x] \subset [y]$. Similarly we can show that $[y] \subset [x]$. Therefore $[x]=[y]$.
– Let $[x]=[y]$. Since $x \in [x]$, then $x \in [y]$. It means that $(y,x) \in R$ and hence $(x,y) \in R$.
(c) says that two classes are either identical or they are disjoint.
Assume that $[x] \neq [y]$. Let us show that then $[x] \cap [y] = \emptyset$.
If $[x] \cap [y] \neq \emptyset$ then there exists $t \in [x] \cap [y]$. Hence, $(x,t) \in R$ and $(y,t) \in R$. We have $(x,t) \in R$ and $(t,y) \in R$. By transitivity of R, $(x,y) \in R$. By part (b), it follows that $[x]=[y]$.
■

Example. $A=\{1,2,3,4\}$, $R=\{(1,1),(2,2),(3,3),(4,4),(3,4),(4,3)\}$
Here $[1]=\{1\}$, $[2]=\{2\}$, $[3]=\{3,4\}=[4]$

Theorem. Let A be a set. Then
(a) any equivalence relation R on A induces a partition of A;

(b) any partition of A gives rise to an equivalence relation R on A.

Proof.
(a) Let R be an equivalence relation R on A. By parts (a) and (c) of the previous Theorem, each element of A belongs to exactly one class of equivalence. These classes of equivalence are blocks of the partition of A induced by R.

(b) Let $\{A_i\}_{i\in I}$ be a partition of A. Define relation R on A as follows:

$$R = \{(x,y) : x \text{ and } y \text{ belong to the same block of partition of } A\}.$$

We have:
- $\forall x \in A \quad \exists i \in I$ s.t. $x \in A_i$. Then $(x,x) \in R$. Hence R is reflexive.
- Let $(x,y) \in R \Rightarrow x$ and y belong to A_j for some $j \Rightarrow y$ and x belong to $A_i \Rightarrow (y,x) \in R$. Hence R is symmetric
- Let $(x,y) \in R$ and $(y,z) \in R$. Then x and y belong to A_j for some j. Hence y and z belong to the same block A_j. Therefore, x and z belong to the same A_j. Hence $(x,z) \in R$, i.e. R is transitive.

Note that $R = (A_1 \times A_1) \cup (A_2 \times A_2) \cup \dots \cup (A_n \times A_n) =$
$$= \bigcup_{i \in I} (A_i \times A_i)$$ ∎

Corollary. For any set A, there is a one-to-one correspondence between the set of equivalence relations on A and the set of partitions of A.

Problem. Let $A \neq \emptyset$, $B \subset A$. Let R be a binary relation on the set of all subsets of A, denoted $\mathcal{P}(A)$, such that

$$(X,Y) \in R \quad \text{if } X, Y \subset A \text{ and } B \cap X = B \cap Y.$$

(a) Show that R is an equivalence relation on $\mathcal{P}(A)$.
(b) If $A = \{1,2,3\}$ and $B = \{1,2\}$, find the partition of $\mathcal{P}(A)$ induced by R.
(c) If $A = \{1,2,3,4,5\}$ and $B = \{1,2,3\}$, find $[X]_R$ if $X = \{1,3,5\}$
HW(d) For $A = \{1,2,3,4,5\}$ and $B = \{1,2,3\}$, how many equivalence classes are in the partition induced by R.

Solution.
(a) $(X,X) \in R$ since $B \cap X = B \cap X \Rightarrow R$ is reflexive

$(X,Y) \in R \Rightarrow B \cap X = B \cap Y \Rightarrow B \cap Y = B \cap X \Rightarrow (Y,X) \in R \Rightarrow$ R is symmetric

$(X,Y) \in R$, $(Y,Z) \in R \Rightarrow B \cap X = B \cap Y = B \cap Z \Rightarrow (X,Z) \in R \Rightarrow$ R is transitive.

(b) $\mathcal{P}(A) = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}, \emptyset\}$

$[\{1\}] = [\{1,3\}]$
$[\{2\}] = [\{2,3\}]$
$[\{3\}] = [\emptyset]$
$[\{1,2\}] = [\{1,2,3\}]$

Partition of $\mathcal{P}(A)$ induced by R is
$A_1 = \{\{1\}, \{1,3\}\}$, $A_2 = \{\{2\}, \{2,3\}\}$, $A_3 = \{\{3\}, \emptyset\}$, $A_4 = \{\{1,2\}, \{1,2,3\}\}$

(c) $X = \{1,3,5\} \in \mathcal{P}(A)$
$[X] = \{Y \in \mathcal{P}(A) : (X,Y) \in R\} = \{Y \in \mathcal{P}(A) : X \cap B = Y \cap B\} =$
$= \{Y \in \mathcal{P}(A) : Y \cap \{1,2,3\} = \{1,3\}\} \Rightarrow$

$\{1,3\}, \{1,3,4\}, \{1,3,5\}, \{1,3,4,5\} \in [X]$.

Product and Sum of Partitions.

Let π_1 and π_2 be two partitions of a set A. Let R_1 and R_2 be the corresponding equivalence relations.

Definition. We say that π_1 is a refinement of π_2, denoted by $\pi_1 \leq \pi_2$, if $R_1 \subset R_2$. In other words, if π_1 is a refinement of π_2 then any two elements that are in the same block of π_1 must also be in the same block of π_2.

Example. $A = \{a, b, c, d, e, f\}$. Let $\pi_1 = \{\{a,b\}, \{c,d,e\}, \{f\}\}$ and $\pi_2 = \{\{a,b,f\}, \{c,d,e\}\}$ be two partitions of A. We have $\pi_1 \leq \pi_2$.
Let us find equivalence relations R_1 and R_2 corresponding to π_1 and π_2. We have

$$R_1 = \{(a,a), (a,b), (b,a), (b,b), (c,d), (c,c), (c,e), (d,c), (d,d), (d,e), (e,c), (e,d), (e,e), (f,f)\}$$

$$R_2 = \{(a,a), (a,b), (a,f), (b,a), (b,b), (b,f), (f,a), (f,b), (f,f), (c,c), (c,d), (c,e), (d,c), (d,d), (d,e), (e,c), (e,d), (e,e)\}.$$

We see that $R_1 \subset R_2$.

Definition. The product of two partitions π_1 and π_2 of a set A, denoted $\pi_1 \circ \pi_2$ is the partition of A corresponding to the equivalence relation $R_1 \cap R_2$.
In other words, $\pi_1 \circ \pi_2$ is the partition of A such that two elements a and b are in the same block of $\pi_1 \circ \pi_2$ if a and b are in the same block of π_1 and also in the same block of π_2.

Example. $A = \{1,2,3,4,5\}$, $\pi_1 = \{\{1,4\}, \{2,3\}, \{5\}\}$, $\pi_2 = \{\{1,4,2\}, \{3,5\}\}$.

$$\pi_1 \circ \pi_2 = \{\{1,4\}, \{2\}, \{3\}, \{5\}\}.$$

$$R_1 = \{(1,1), (1,4), (4,1), (4,4), (2,2), (2,3), (3,2), (3,3), (5,5)\}$$

$$R_2 = \{(1,1), (1,4), (1,2), (4,1), (4,4), (4,2), (2,1), (2,4), (2,2), (3,3), (3,5), (5,3), (5,$$

$$R_1 \cap R_2 = \{(1,1), (1,4), (4,1), (4,4), (2,2), (3,3), (5,5)\}$$

The partition corresponding to $R_1 \cap R_2$ is clearly $\pi_1 \circ \pi_2$.

To define the sum of two partition, we need one additional definition.

Definition. Let R be a binary relation on A. The transitive extension of R, denoted R_1, is a binary relation on A such that $R \subset R_1$ and moreover, if $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R_1$.

Example. $A = \{a, b, c, d\}$

R	a	b	c	d
a		✓		
b			✓	
c		✓		✓
d				

R_1	a	b	c	d
a		✓	✓	
b		✓	✓	✓
c		✓	✓	✓
d				

1) Note that R_1 is not necessarily transitive.

2) Note that if R is transitive then $R = R_1$.

Definition. Let R_2 denote the transitive extension of R_1, and, in general, let R_{i+1} denote the transitive extension of R_i. The transitive closure of R, denoted R^*, is $\bigcup_i R_i$.

Example. Let A and R be the same as in the previous example.

R_2	a	b	c	d
a		✓	✓	✓
b		✓	✓	✓
c		✓	✓	✓
d				

Since R_2 is transitive then

$R_3 = R_2$, $R_4 = R_2$, ..., $R_n = R_2$ $\forall n \geq 2$.

Therefore $R^* = R_2 \cup R_1 = R_2$

Definition. The sum of π_1 and π_2, denoted $\pi_1 + \pi_2$, is the partition corresponding to the equivalence relation $(R_1 \cup R_2)^*$.