

## Repetitions

Last week we counted the number of ways of putting  $r$  balls into  $n$  boxes with at most one ball to a box.

Suppose we allow any number of balls in a box. If the balls are all colored differently, then there are  $n$  choices for the first ball,  $n$  for the second, and so forth. There are  $n^r$  possibilities. When the balls are all the same color, however, we expect far fewer possibilities.

Problem 1. How many possibilities do we have to put 3 white balls into ten boxes if we are allowed to put as many balls into a box as we like.

Solution.

There are 3 possibilities: (a) each ball goes into a different box; (b) two balls go into the same box but the third goes into its own box; (c) all the balls go into the same box.

(a) can be done in  $C(10,3)$  different ways

(b) can be done in  $10 \times 9$  ways

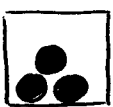
(c) can be done in 10 ways.

Therefore, totally there are  $C(10,3) + 90 + 10 = 220$  possibilities to place three balls into 10 numbered boxes.

Problem 2. In how many ways can ten balls, all of the same color, be put into three boxes?

Solution. While this problem is obviously similar to the previous, this time it is very difficult to do a case-by-case examination of the possibilities.

One way to put the balls into the boxes is the following



Box 1



Box 2



Box 3

This situation can be represented by the 12-digit string

000100000100

There is a one-to-one correspondence between the ways of putting the balls into boxes and 12-digit

strings consisting of two 1's and ten 0's.

The string 001000000100 corresponds to 

To count the number of ways of putting ten balls in three boxes is just to count the number of 12 digit 0-1 strings which contain exactly two 1's. This number is easy to find.

Answer is  $C(12, 2)$ .

Theorem. The number of ways to put  $r$  identical balls into  $n$  boxes is

$$C(n+r-1, r) = \frac{(n+r-1)!}{r!(n-1)!}$$

Problem 3. Doughnuts come in 30 different varieties and Catherine wants to buy a dozen. How many choices does she have?

Solution. Imagine that the 30 varieties are in  $n=30$  boxes labeled "d1" "d2" ..., "d30" Catherine can indicate her choice by dropping  $r=12$  identical balls into the boxes. So there are

$$C(30+12-1, 12) = C(41, 12) \text{ possibilities.}$$

Problem 4. Catherine wants to buy 30 doughnuts and finds just 12 varieties available. In how many ways can she make her selection?

Answer:  $C(41, 30)$ .

Remark. The problem of selecting  $r$  objects from  $n$  distinct objects allowing repeated selections, can be viewed as placing  $r$  balls of the same color into  $n$  numbered boxes. Therefore, the number of ways to select  $r$  objects (with repetitions allowed) from  $n$  distinct objects, is  $C(n+r-1, r)$ .

Problem 5. The number of ways to choose three out of seven days (with repetitions allowed) is  $C(7+3-1, 3)$

The number of ways to choose seven out of three days (with repetitions necessarily allowed) is  $C(7+3-1, 7)$ .

Problem 6. How many  $n$ -digit decimal numbers have their digits in nondecreasing order? (Note that the first digit of an  $n$ -digit number must not be 0).

Solution. Note that we do not meet 0's in such numbers.

if  $n=3$ , an example of such a number is 113.

if  $n=4$ , ——— " ——— 3448

In general, there is a one-to-one correspondence between such numbers and strings of 0's and 1's containing  $n+8$  elements,  $n$  of which are 0.

For example,  $n=4$ , 3448  $\leftrightarrow$  110100111101

$\uparrow$   $\uparrow$   $\uparrow$   $\uparrow$   $\uparrow$   $\uparrow$   $\uparrow$   $\uparrow$   $\uparrow$   $\uparrow$   
 zero ones zero twos two fours zero 5,6,7 and 9

$$\text{Answer is } C(n+9-1, n) = C(n+8, n).$$

## Discrete Probability

Recall that an experiment is a physical process that has a number of observable outcomes.

Definition. The set of all possible outcomes of an experiment is called the sample space of the experiment. Elements of the sample space (outcomes) are called samples (or sample points)

Example. 1) for the experiment of tossing a coin, the sample space is  $S = \{h, t\}$  (head, tail)  
 2) the sample space of tossing two coins, is  $S = \{hh, ht, th, tt\}$ .

With each sample in a sample space we associate a real number called the probability of that sample.

Let  $x_i$  be a sample,  $p(x_i)$  a probability associated with  $x_i$ . The probabilities associated with the samples must satisfy two conditions:

- 1) The probability of each sample is a nonnegative number  $\leq 1$ . ( $\forall x_i \in S, 0 \leq p(x_i) \leq 1$ )
- 2) The sum of the probabilities of all the samples in the sample space is equal to 1. ( $\sum_{x_i \in S} p(x_i) = 1$ )

Remark. A sample with a larger probability is more likely to take place, while a sample with a smaller probability is less likely to take place.

Ex. For the experiment of tossing two coins with the sample space  $S = \{hh, ht, th, tt\}$ , we might have

$$p(hh) = \frac{1}{4}, \quad p(ht) = \frac{1}{4}, \quad p(th) = \frac{1}{4}, \quad p(tt) = \frac{1}{4}$$

if the coin is fair.

Definition. An event is a subset of the outcomes of an experiment. An event is said to occur if any one of the samples in the event occurs.

Example. When we roll a die, getting a 1 is an event, getting an even number (2, 4, 6) is another event.

Definition. An event that contains one sample is called a simple event. An event that contains more than one sample is called a compound event.

The probability of occurrence of an event is defined as the sum of probabilities of the samples in the subset.

Example 1. For the experiment of rolling a die, the sample space consists of six samples. If we suppose the probability of occurrence of each of these samples is  $\frac{1}{6}$  then the probability of getting an even number is  $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$ .

Example 2. Confirm the observation that out of 23 people the chance is less than 50-50 that no two of them will have the same birthday.

Solution. Consider the sample space consisting of  $366^{23}$  samples corresponding to all possible distributions of birthday of 23 people. Let us assume that distributions are equiprobable. Out of  $366^{23}$  samples,  $P(366, 23)$  correspond to distributions of birthdays such that no two of the 23 people have the same birthday.

The answer is 
$$\frac{P(366, 23)}{366^{23}} = 0.494$$

Example 3. Eight students are standing in line for an interview. Determine the probability that there are exactly two 1-year und. students, two 2nd-year und. students, two 3rd-year und. students, two 4th-year und. students in the line.

Solution. The sample space consists of  $4^8$  samples corresponding to all possible ways of classes the students are from. Let us assume that these are equiprobable samples. There are  $\frac{8!}{2!2!2!2!}$  samples corresponding to the case in which there are two students from each class. Thus, the probability is 
$$\frac{8!}{2!2!2!2!4^8} = 0.0385.$$

## Binary relations.

Definition. The cartesian product of  $A$  and  $B$ , denoted  $A \times B$ , is the set of all ordered pairs of the form  $(a, b)$  where  $a \in A$  and  $b \in B$ .

Example.  $\{1, 2\} \times \{3, 1, 4\} = \{(1, 3), (1, 1), (1, 4), (2, 3), (2, 1), (2, 4)\}$ ,  
 $\{2, 4\} \times \{2, 4\} = \{(2, 4), (2, 2), (4, 2), (4, 4)\}$ .

Definition. A binary relation from  $A$  to  $B$  is a subset of  $A \times B$ .

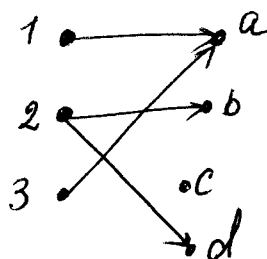
Example. Let  $A = \{a, b, c, d\}$ ,  $B = \{b, d\}$   
 $R_1 = \{(b, b), (c, d)\}$   
 $R_2 = \{(a, d), (a, b), (b, b), (b, d)\}$  } are binary relations from  $A$  to  $B$ .

Besides a list of the ordered pairs, a binary relation can also be represented by tabular form or graphical form.

Let  $A = \{1, 2, 3\}$ ,  $B = \{a, b, c, d\}$ ,  $R = \{(1, a), (2, b), (3, a), (2, d)\}$ .

	a	b	c	d
1	✓			
2		✓		✓
3	✓			

rows correspond to the elements of  $A$ , columns correspond to the elements of  $B$ , a check mark in a cell means the cell is related to  $R$



elements of  $A$  are in the left-hand column, elements of  $B$  are in the right-hand column, an arrow from the left-hand column to the right-hand column indicates the corresponding pair in  $R$ .

Let  $R_1$  and  $R_2$  be two binary relations from  $A$  to  $B$ . Then  $R_1$  and  $R_2$  are subsets of  $A \times B$ . Therefore, we can consider  $R_1 \cup R_2$ ,  $R_1 \cap R_2$ ,  $R_1 \setminus R_2$ , the symmetric difference

$$R_1 \oplus R_2 := (R_1 \cup R_2) \setminus (R_1 \cap R_2).$$

Example. Let  $A = \{a, b, c, d\}$  be a set of students,  
 $B = \{M101, M102, M110, M132\}$  be a set of courses.  
 Let  $R_1$  be a binary relation from  $A$  to  $B$  describing  
 the courses the students are taking and a binary  
 relation  $R_2$  describe the courses students are  
 interested in.

$R_1$	M101	M102	M110	M132
a	✓		✓	
b		✓		✓
c	✓			✓
d		✓		

$R_2$	M101	M102	M110	M132
a		✓	✓	
b		✓	✓	
c	✓	✓		✓
d				✓

- $R_1 \cup R_2$  describes the courses the students are either taking or are interested in.
- $R_1 \cap R_2 = \{(a, M110), (b, M102), (c, M101), (c, M132)\}$  describes the courses the students are taking and are also interested in.
- $R_1 \setminus R_2 = \{(a, M101), (b, M132), (d, M102)\}$  describes the courses the students are taking but not interested in.
- $(R_1 \cup R_2) \setminus (R_1 \cap R_2) = R_1 \oplus R_2 = \{(a, M101), (a, M102), (b, M110), (b, M132), (c, M102), (d, M102), (d, M132)\}$  describes the courses the students are interested in but not taking or are taking but not interested in.

Definition. A binary relation from  $A$  to  $A$  is called a binary relation on  $A$ .

Example. Let  $A = \mathbb{N} = \{1, 2, 3, \dots\}$ ,  $R$  be a binary relation on  $A$  such that  $(a, b)$  is in  $R$  if  $-a + b \geq 0$ .  
 We have  $(1, 1) \in R$ ,  $(3, 7) \in R$ ,  $(5, 4) \notin R$ .

## Properties of binary relations.

Definition. A binary relation  $R$  on set  $A$  is called a reflexive relation if  $(a, a) \in R$  for every  $a \in R$ .

A binary relation  $R$  on set  $A$  is called a symmetric relation if  $(a, b) \in R$  implies  $(b, a) \in R$ .

A binary relation  $R$  on set  $A$  is called a transitive relation if  $(a, c) \in R$  whenever  $(a, b)$  and  $(b, c)$  are in  $R$ .

Example. Let  $R$  be a binary relation on the set of all strings of 0's and 1's such that  $R = \{(a, b) : a \text{ and } b \text{ are strings that have the same number of 0's}\}$ . (a) Is  $R$  reflexive? (b) Symmetric? (c) Transitive?

Solution.

(a) Yes, (b) Yes, (c) Yes.

Example. Let  $R$  be a binary relation on the set of all positive integers. Is it reflexive, symmetric, transitive if

(a)  $R = \{(a, b) : a - b \text{ is an even positive integer}\}$

(b)  $R = \{(a, b) : a = b^2\}$ .

Solution:

(a) -  $(a, a) \notin R$ , since  $a - a = 0$  is not positive integer  
Then  $R$  is not reflexive

- if  $(a, b) \in R \Rightarrow a - b$  is pos. integer  $\Rightarrow b - a$  is not positive integer  $\Rightarrow (b, a) \notin R$ . Then  $R$  is not symmetric

- if  $(a, b) \in R, (b, c) \in R \Rightarrow a - b$  and  $b - c$  are positive even integers  $\Rightarrow (a - b) + (b - c) = a - c$  is even positive integer  $\Rightarrow (a, c) \in R$ . Then  $R$  is transitive.

(b) -  $(a, a) \notin R$ , since  $a \neq a^2$  for all positive integers.  
Then  $R$  is not reflexive

-  $(a, b) \in R \Rightarrow a = b^2 \Rightarrow b \neq a^2$  for all positive integers s.t.  $a = b^2 \Rightarrow (b, a) \notin R$ . Then  $R$  is not symmetric.

-  $(a, b) \in R, (b, c) \in R \Rightarrow a = b^2, b = c^2 \Rightarrow a = c^4 \neq c^2$   
Then  $(a, c) \notin R$ . Then  $R$  is not transitive.



Remark. Let a binary relation on a set is represented in tabular form.

- (a) A binary relation is reflexive if and only if all the cells on the main diagonal of the table contain check marks.
- (b) A binary relation on a set is symmetric  $\Leftrightarrow$  the check marks are in cells that are symmetrical with respect to the main diagonal

$R_1$	a	b	c	d
a	✓			✓
b	✓	✓		✓
c	✓		✓	
d		✓		✓

$R_2$	a	b	c	d
a	✓	✓		✓
b	✓			✓
c			✓	
d	✓	✓		✓

$R_1$  is reflexive  
 $R_1$  is not symmetric  
 $(a, b) \notin R_1, (b, a) \in R_1$

$R_2$  is not reflexive  
 $R_2$  is symmetric.

Problem. Let  $A$  be a set with 10 distinct elements.

- (a) How many different binary relations on  $A$  are there?  
 (b) How many of them are reflexive?  
 (c) How many of them are symmetric?  
 (d) How many of them are reflexive and symmetric?

Solution.

(a) to answer (a), we have to count the number of subsets of  $A \times A$ . Since  $A \times A$  has 100 different elements then the number of subsets of  $A \times A$  is  $2^{100}$ . So, there are  $2^{100}$  binary relations on  $A$ .

(b) let  $A = \{1, 2, 3, \dots, 10\}$ . Each reflexive relation must have  $(1, 1), (2, 2), \dots, (10, 10)$  as elements. We are left with 90 other elements that can be in a reflexive relation or not. So, there are  $2^{90}$  reflexive relations on  $A$ .

(c)

	1	2	3	4	5	6	7	8	9	10
1										
2										
3										
4										
5										
6										
7										
8										
9										
10										

Let  $R$  be a symmetric relation on  $A$ .

Let  $(a, b) \in R$ ,  $a \leq b$ .

Then  $(b, a) \in R$ .

There are  $10+9+8+\dots+1=55$  different elements in  $A \times A$  such that  $a \leq b$ .

Therefore, there are  $2^{55}$  symmetric relations.

(d) Any symmetric and reflexive binary relation on  $A$  must include  $(1,1), (2,2), \dots, (10,10)$ .

Therefore there are  $55-10$  elements that can be in sym. and reflexive relation and be of the kind  $(a,b)$ ,  $a < b$ .

Thus, there are  $2^{45}$  different binary relations on  $A$  that are reflexive and symmetric.

A binary relation may have none, one or more of the following properties: reflexivity, symmetry, transitivity.

Definition. A binary relation on a set is said to be an equivalence relation if it is reflexive, symmetric and transitive.

HW Problem. Let  $A$  be a set of students and  $R$  be a binary relation on  $A$  such that  $(a, b) \in R$  if and only if  $a$  lives in the same dormitory as  $b$ . Show that  $R$  is an equivalence relation.