

# Methods of Proof

Week 3.

Definition. A theorem is a mathematical proposition that is true.

Many theorems are conditional propositions. For example,

Theorem. If  $f(x)$  and  $g(x)$  are continuous functions then  $f(x) \pm g(x)$  are continuous functions.

Most theorems that are not conditional propositions can be rewritten in the form of conditional proposition. For example,

Theorem. A set with  $n$  elements has exactly  $2^n$  subsets. can be written in the form

Theorem. If  $S$  is a set with  $n$  elements then  $S$  has exactly  $2^n$  subsets.

Definition. If Theorem is in the form "if  $p$  then  $q$ " then  $p$  is called the hypothesis and  $q$  is called the conclusion of the theorem.

By a proof of a theorem we mean a logical argument that establishes the theorem to be true. Suppose we want to prove the theorem;  $p \rightarrow q$ . Since  $p \rightarrow q$  is true whenever  $p$  is false, we need only show that whenever  $p$  is true, so is  $q$ .

We will consider the following methods of Proof;

- Direct Proof
- Proof by Cases
- Proof by Contrapositive
- Proof by Contradiction.

(1) Direct Proof. In a direct proof we assume that the hypothesis of the theorem,  $p$ , is true and demonstrate that the conclusion,  $q$ , is true. Then  $p \rightarrow q$  is true.

Theorem. If  $n$  is an even integer then  $n^2$  is an even integer.

Proof. Assume  $n$  is an even. Then  $n = 2k$  for some integer  $k$ . Then

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2).$$

Hence,  $n^2$  can be expressed as 2 times the integer  $2k^2$  and so  $n^2$  is even. ■

(2) Proof by Cases. Sometimes a direct proof is made simpler by breaking it into a number of cases, one of which must hold and each of which leads to the desired conclusion.

Theorem. If  $n$  is an integer then  $9n^2 + 3n - 2$  is even.

Proof. Case 1.  $n$  is even.

Since  $n$  is even then  $9n^2$  and  $3n$  are even too. Thus  $9n^2 + 3n - 2$  is even because it is the sum of three even integers.

Case 2.  $n$  is odd.

The product of odd integers is odd. In this case, since  $n$  is odd,  $9n^2$  and  $3n$  are also odd. The sum of two odd (numbers) integers is even. Thus  $9n^2 + 3n$  is even. So  $9n^2 + 3n - 2$  is even. ■

(3) Proof by Contrapositive. We know that propositions " $p \rightarrow q$ " and " $\bar{q} \rightarrow \bar{p}$ " are equivalent. It means that if we prove that " $\bar{q} \rightarrow \bar{p}$ " is true then immediately " $p \rightarrow q$ " is true.

Theorem. If  $x + y > 100$  then  $x > 50$  or  $y > 50$ .

Proof. Let  $p$ :  $x + y > 100$ ;  $q$ :  $x > 50$  or  $y > 50$ . Instead of proving  $p \rightarrow q$  we will prove  $\bar{q} \rightarrow \bar{p}$ .

Assume  $x \leq 50$  and  $y \leq 50$ , i.e.  $\bar{q}$  is true.

Then  $x + y \leq 50 + 50 = 100$ , i.e.  $\bar{p}$  is true. We proved  $\bar{q} \rightarrow \bar{p}$ , or the same  $p \rightarrow q$ . ■

(4) Proof by Contradiction. Sometimes a direct proof of a statement  $p \rightarrow q$  seems hopeless. We simply do not know how to begin. In this case, we can sometimes make progress by assuming that proposition  $p \rightarrow q$  is false, i.e.  $p$  is true and  $q$  is false. If this assumption leads to a proposition which is obviously false or to a proposition which contradicts something else, then we will conclude that our assumption is wrong, i.e. " $p \rightarrow q$  is false" is wrong, i.e.  $p \rightarrow q$  is true.

Theorem. If  $a$  is a nonzero rational number and  $b$  is an irrational number then  $ab$  is irrational.

Proof. Assume that statement of the theorem is wrong. Then  $a$  is a nonzero rational number,  $b$  is an irrational number and  $ab$  is rational, i.e.

$$a = \frac{k}{l}, \quad ab = \frac{m}{n} \quad \text{for some integers } k, l, m, n.$$

Thus  $b = \frac{m}{na} = \frac{ml}{nk}$  with  $nk \neq 0$ . So  $b$  is rational.

This contradicts the fact that  $b$  is irrational. So the statement of the theorem is correct. ■

## The Rules of Sum and Product.

By an experiment, we mean a physical process that has a number of observable outcomes.

For example, -placing 1 red and 1 blue balls in 3 boxes  
 -tossing a coin  
 can be considered as experiments.

When we consider the outcomes of several experiments, we shall follow the following rules:

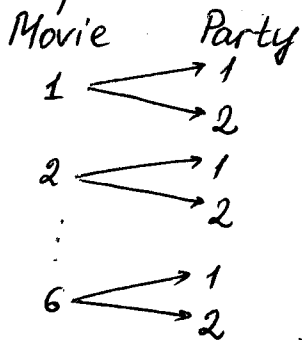
Rule of Product. If one experiment has  $m$  possible outcomes and another experiment has  $n$  possible outcomes, then there are  $m \times n$  possible outcomes when both of these experiments take place.

Rule of Sum. If one experiment has  $m$  possible outcomes and another experiment has  $n$  possible outcomes, then there are  $m+n$  possible outcomes when exactly one of these experiments takes place.

Example 1. It is Friday night and Ursula has been invited to two parties but feels more inclined to go to a movie. There are six new movies in town. How many possible ways does Ursula have (to spend her evening) assuming that she is not permitted to have two engagements in the same evening?

Answer. By Rule of Sum, Ursula has  $2+6=8$  different possibilities to spend her evening.

Example 2. Assume Ursula's parents had a change of mind and permitted her to go to a party after a movie. Then by the Rule of Product, Ursula has  $2 \times 6 = 12$  ways in which she could spend her Friday night.



} There are  $6 \times 2 = 12$  ways to choose one of six movies and then one of two parties.

Example 3. License plates in Canada consist of four letters followed by three of the digits 0-9 (not necessarily distinct). How many different license plates can be made in Canada?

Solution.

There are 26 ways in which the first letter can be chosen, similarly for the third, second and fourth.

By the multiplication Rule, the number of ways in which the letters can be chosen is  $26 \times 26 \times 26 \times 26 = 26^4$ .

By the same reasoning there are  $10^3$  ways in which the final three digits can be selected.

Therefore, there are  $26^4 \times 10^3 = 456\,976\,000$  different license plates (different).

## Permutations

Problem. The registrar of a school must schedule six examinations in an eight-day period and has promised the students not to schedule more than one examination per day. How many different schedules can be made?

Solution.

The first exam can be given on any of the eight days, but once this day has been settled, the second exam can be given on only one of the remaining seven days. There are  $8 \times 7 = 56$  pairs of days on which the first two examinations can be scheduled. Once the first two exams has been settled, there remain six days on which to schedule the third exam. Continuing to reason this way, we see that there are  $8 \times 7 \times 6 \times 5 \times 4 \times 3 = 20\,160$  possible schedules.

The solution to a counting problem often involves the product of consecutive integers. We use the following notation for integers  $n$  and  $r$ ,  $1 \leq r \leq n$ ,

$$P(n, r) = n(n-1)(n-2) \dots (n-r+1) = \frac{n!}{(n-r)!},$$

where  $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$

Problem. In how many ways can three balls colored red, blue and white be placed in 10 boxes numbered 1, 2, 3, ..., 10, if each box can hold only one ball.

Solution: the number of ways  $10 \times 9 \times 8 = P(10, 3)$ . For us, the symbol  $P(n, r)$  is a device which makes it easy to write down the answers to certain counting

problems.

Definition. A permutation of a set of distinct symbols (objects) is an arrangement of them in a line in some order.

- Example: 1)  $ab$  and  $ba$  are permutations of the symbols  $a$  and  $b$ .
- 2)  $1642, 4126, 6241$  are permutations of  $1, 2, 4$  and  $6$ .
- 3)  $1642, 4126, 6241$  are also examples of 4-permutations of the symbols  $1, 2, 3, 4, 5, 6$ , that is permutations of  $1, 2, 3, 4, 5, 6$  taken four at a time.

Definition. For natural numbers  $r$  and  $n, r \leq n$ , an  $r$ -permutation of  $n$  symbols is a permutation of  $r$  of them.

Theorem. The number of permutations of  $n$  symbols is  $n!$ . The number of  $r$ -permutations of  $n$  symbols is  $P(n, r)$ .

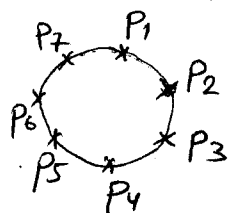
Proof. Assume that we have  $n$  symbols  $a_1, a_2, \dots, a_n$ . First symbol  $a_1$  can be put on any of  $n$  places, then  $a_2$  can be placed on any of  $(n-1)$  remaining places and so on.

Therefore, we have  $n(n-1)(n-2) \dots 2 \cdot 1 = n!$  permutations of a set of  $n$  symbols.

The second statement can be proved similarly. ■

Problem. There are  $7! = 5040$  ways in which seven people can form a line. In how many ways can seven people form a circle?

Solution.



A circle is determined by the order of the people to the right of any one of the individuals, say  $P_1$ . There are 6 possibilities for the person on  $P_1$ 's right then 5 possibilities for the next person,

four for the next and so on. The number of possible circles is  $6! = 720$ .

Problem. In how many ways can ten adults and five children stand in a line so that no two children are next to each other?

Solution. Assume that we have ten adults named A, B, C, D, E, F, G, H, I, J.

Consider the following line

$$x \times J \times I \times H \times G \times F \times E \times D \times C \times B \times A \times x$$

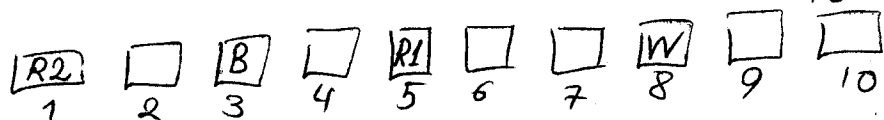
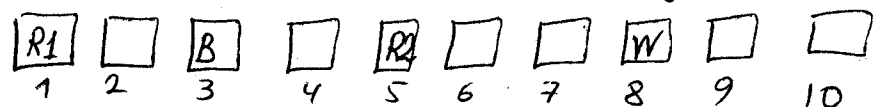
In this line x's represents the 11 possible locations for the children. For each such line, the first child can be positioned in any of the 11 x spots, the second child in any of the remaining 10, and so on. Hence, the children can be positioned in  $11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = P(11, 5)$  ways. For each such positioning, there are  $10!$  ways of ordering the adults A, B, ..., J. By the Rule of Product, the number of lines and adults is  $10! \cdot P(11, 5)$ .

HW Problem. In how many ways can ten adults and five children stand in a circle so that no two children are next to each other?  
(Answer:  $9! \cdot P(10, 5)$ .)

Problem. In how many ways four balls: two red, one blue, one white, can be placed into 10 numbered boxes, if each box can hold only one ball.

Solution: Among balls we have two red indistinguishable balls. If we distinguish them by writing R1 on one and R2 on the other then there will be  $P(10, 4) = 5040$  ways of placing R1, R2, blue, white balls into 10 numbered boxes.

Consider such two placings



If we do not distinguish red balls then these two placements are the same. So in this case, two ways of placements actually become one. Indeed, the 5040 placements can be paired off in a similar way so that every pair of placements becomes one when we do not distinguish red balls.

Consequently, there are  $\frac{P(10,4)}{2} = 2520$  ways to place two red, one blue and one white ball in 10 numbered boxes.

Theorem: The number of ways to place  $r$  colored balls in  $n$  numbered boxes, where  $q_1$  of these balls are of one color,  $q_2$  of them of a second color, ..., and  $q_t$  of them are of  $t$ th color, is

$$\frac{P(n, r)}{q_1! q_2! \dots q_t!}$$

HW Problem. Find the number of different messages that can be represented by sequences of three dashes (-) and two dots (·).

(Answer:  $\frac{5!}{3!2!} = 10$ ) ,  $P(5,5) = \frac{5!}{0!} = 5!$

### Combinations

Definition. A combination of a set of objects is a subset of them. A subset of  $r$  objects is called an  $r$ -combination of the objects taken  $r$  at a time.

Theorem: Let  $n$  and  $r$  be integers with  $n \geq 0$  and  $0 \leq r \leq n$ . The number of ways to choose  $r$  objects from  $n$  is  $\frac{n!}{r!(n-r)!} =: C(n, r) = \binom{n}{r}$ .



Proof. Agreement:  $0! = 1$

Let  $N$  be the number of ways to choose  $r$  objects from  $n$  given objects. Notice that for each way of choosing  $r$  objects, there are  $r!$  ways to order them. By the multiplication rule,

$$P(n, r) = N \times r!$$

Therefore,

$$N = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!} = C(n, r). \quad \blacksquare$$

Problem. In how many ways spaghetti dinners can be scheduled three times each week.

Solution.

The number of scheduling is  $C(7, 3) = \frac{7!}{3!(7-3)!} = \frac{7!}{3!4!} = 35$

Remark 1. Theorem can be rewritten:

The number of  $r$ -combinations of  $n$  objects is  $C(n, r) = \frac{n!}{r!(n-r)!}$

Remark 2. The distinction between permutation and combination is the distinction between order and selection.

Box1, Box2, Box is one combination of boxes;

Box2, Box1, Box4 is the same combination, but a different permutation. A permutation takes order into account; a combination involves only selection.

Problem. In how many ways can 20 students out of a class of 32 be chosen to attend class on a Thursday afternoon (and take notes for the others) if

(a) Paul refuses to go to class?

(b) Michelle insists on going?

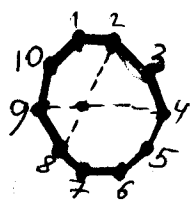
Solution:

(a) The answer is  $C(31, 20)$  since it is necessary to select 20 students from the 31 students excluding Paul.

(b) The number of possibilities is  $C(31, 19)$  since 19 students must be chosen from 31.

Problem. If no three diagonals of a convex decagon meet at the same point inside the decagon, into how many line segments are the diagonals divided by their intersections?

Solution:



1) Let us find the number of diagonals.

There are  $C(10, 2)$  straight lines joining 10 vertices of the decagon. Among these  $C(10, 2)$  there are 10 sides of the decagon. Therefore there are

$$C(10, 2) - 10 = 45 - 10 = 35 \text{ diagonals.}$$

2) Let us find the number of intersection between the diagonals.

Since for every four vertices we can count exactly one intersection between the diagonals then there are a total of  $C(10, 4) = 210$  intersections between the diagonals.

3) If on a diagonal there are  $k$  intersecting points then it is divided by points into  $(k+1)$  straight line segments.

Therefore, if we have  $k_1$  points on the first diagonal  
 $k_2$  " " " " second " " "  
 $\vdots$   
 $k_{35}$  " " " " 35th " " "

then there are  $(k_1 + 1) + (k_2 + 1) + \dots + (k_{35} + 1) =$   
 $= k_1 + k_2 + \dots + k_{35} + 35$  straight line segments.

Since each intersecting point lies on two diagonals the total number of straight-line segments into which the diagonals are divided is  $35 + 2 \times 210 = 455$ .