

## Propositions

Definition: A proposition is a declarative sentence that is either true or false.

For example, "Yesterday the weather was nice"

"I will draw this evening"

"Children like to play games"

are all propositions. On the other hand,

"What time is it?", "What a beautiful evening!", "Study hard" are not propositions because they are not declarative sentences, and consequently, it is not meaningful to speak of them being true or false.

We will refer to propositions by symbolic names. For example, let  $p$  denote the proposition "John likes sweets". We say that  $p$  has value  $T$  if the proposition is true and we say that the value of  $p$  is  $F$  if the proposition  $p$  is false.

Definition: We say that two propositions  $p$  and  $q$  are equivalent if

when  $p$  is true then  $q$  is true and conversely, if  $q$  is true then  $p$  is true;

when  $p$  is false then  $q$  is false and if  $q$  is false then  $p$  is false.

For example:

① Let  $p$ : "Water froze this morning"

$q$ : "Temperature was below  $0^{\circ}\text{C}$  this morning"

Then  $p$  and  $q$  are equivalent propositions.

② Let  $p$ : " $x$  is such a real number that  $x^2 \geq 1$ ".

$q$ : " $x$  is a real number larger or equal to 1"

Since  $p$  follows from  $q$  but  $q$  does not follow from  $p$  then  $p$  and  $q$  are not equivalent propositions.

Propositions can be combined to yield new propositions. We will study propositions formed from other propositions using the following expressions that are called connectives: and, or, not, "...".

(a) The conjunction of two statements is formed by joining the statements with the word "and". For example, the conjunction of the propositions

$p$ : Today is Monday      and       $q$ : I went to school  
is the proposition

$p \wedge q$ : Today is Monday, and I went to school.

The proposition  $p \wedge q$  is true only when both of the original propositions  $p$  and  $q$  are true. Thus, the truth table for the connective "and" is the following

$p$	$q$	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

(b) The disjunction of two propositions is formed by joining the propositions with the word "or".

Let  $p$  and  $q$  be propositions from the previous example.  
 $p \vee q$ : "Today is Monday, or I went to school".

This statement is true if at least one of the original propositions  $p$  and  $q$  is true. The truth table for the connective "or" is the following

$p$	$q$	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

(c) Let  $p$  be a proposition. We define the negation of  $p$ , denoted  $\bar{p}$ , to be a proposition which is true when  $p$  is false, and is false when  $p$  is true.

Let  $p$ : "All doctors are rich", then  $\bar{p}$ : "Some doctors are not rich".

$q$ : "Some students do not pass calculus".

Then  $\bar{q}$ : "All students pass calculus".

The truth table is the following

Week 2

P	$\bar{P}$
T	F
F	T

We will also join two propositions with two other connectives such as "if ... then ..." and "if and only if".

④ A proposition containing the connective "if ... then ..." is called a conditional proposition. For example, suppose Mary is a student we know, and  $p$  and  $q$  are the statements:

$p$ : Mary was at the play Thursday night

$q$ : Mary doesn't have an 8 o'clock class on Friday morning.

Then the conditional "if  $p$  then  $q$ ", or the same " $p \rightarrow q$ " is the proposition

"If Mary was at the play Thursday night, then she doesn't have an 8 o'clock class on Friday morning."

(Note:  $p \rightarrow q$  can be read as "p implies q").

Remark: It is important to understand that conditional proposition should not be interpreted in terms of cause and effect.  $p \rightarrow q$  means if  $p$  is true then  $q$  is true

Example. Let  $p$ : There is a snowfall of 10sm or more on Feb. 17, 2002.  
 $q$ : Cars are not parked overnight on city streets on Feb 17, 2002.

Assume there is a city order designed to aid city street crews in removing snow. The order states:  
 If there is a snowfall of 10 or more sm, then cars can not be parked overnight on city streets.

Let us consider under what circumstances the conditional statement  $p \rightarrow q$  is false (the order has been violated)

P	q	$P \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

It may seem unnatural to regard a conditional proposition  $p \rightarrow q$  as being true whenever  $p$  is false. It seems reasonable to regard a cond. proposition as being not applicable when  $p$  is false. But then  $p \rightarrow q$  would be neither true, nor false when  $p$  is false. So,  $p \rightarrow q$  would not be a proposition by our definition. For this reason

logicians consider a conditional proposition  $\overset{p \rightarrow q}{\vee}$  to be true if  $p$  is false.

- ⑤ The biconditional proposition  $p \leftrightarrow q$  means  $p \rightarrow q$  and  $q \rightarrow p$ . We read the biconditional statement  $p \leftrightarrow q$  as "p if and only if q", or "p is necessary and sufficient for q". For example,  
 Mary was at the play Thursday night if and only if she doesn't have a class at 8:00 on Friday morning.

Truth table.

$p$	$q$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Problem An island has two tribes of natives. Any native from the first tribe always tells the truth, while any native from the other tribe always lies. You arrive at the island and ask a native is there is gold on the island. He answers, "There is gold on the island if and only if I always tell the truth." Is there gold on the island?

Solution:

Let  $p$ : "The person tells the truth"  
 $q$ : "there is gold on the island"

Thus the persons answer is " $p \leftrightarrow q$ "

Assume  $p$  is true then  $p \leftrightarrow q$  is true. By truth table for  $p \leftrightarrow q$ , we have  $q$  is true.

Assume  $p$  is false then " $p \leftrightarrow q$ " is false, then  $q$  is true. Thus in both cases we can conclude that there is gold on the island, although the native could be from either tribe.

Problem 2: Show that propositions  $p \rightarrow q$  and  $\bar{q} \rightarrow \bar{p}$  are equivalent

$p$	$q$	$\bar{p}$	$\bar{q}$	$p \rightarrow q$	$\bar{q} \rightarrow \bar{p}$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

We see that if  $p \rightarrow q$  is True then  $\bar{q} \rightarrow \bar{p}$  is true and conversely and if  $p \rightarrow q$  is false then  $\bar{q} \rightarrow \bar{p}$  is false, i.e.  $p \rightarrow q$  and  $\bar{q} \rightarrow \bar{p}$  are equivalent.

Sometimes, the proposition  $\bar{q} \rightarrow \bar{p}$  is called the contrapositive of  $p \rightarrow q$

Example: From the statement: "If it isn't raining today, then I am going to the beach today", form the contrapositive statement.

Solution:

The desired contrapositive statement is

If I am not going to the beach today then it is raining today.

Problem Show that propositions  $\overline{p \vee q}$  and  $\bar{p} \wedge \bar{q}$  are equivalent.

Proof: When analysing a complicated proposition involving connectives, it is often useful to consider the simpler propositions that form it. The truth or falsity of the complicated statement can be (considered) determined by considering the truth or falsity of the simpler statements.

$p$	$q$	$\bar{p}$	$\bar{q}$	$p \vee q$	$\overline{p \vee q}$	$\bar{p} \wedge \bar{q}$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Problem.

Professor Lai has just returned from a visit to an island where each inhabitant either always tells the truth or always lies. He told us that he heard the following statements made by two of the island's inhabitants A and B:

A: B always lies

B: A always tells the truth.

What can you say about Prof. Lai?

Solution:

There are four possibilities:

- |                                   |                            |                            |                     |
|-----------------------------------|----------------------------|----------------------------|---------------------|
| 1) A tells truth<br>B tells truth | 2) A tells truth<br>B lies | 3) A lies<br>B tells truth | 4) A lies<br>B lies |
|-----------------------------------|----------------------------|----------------------------|---------------------|

In 1),

A tells the truth  $\Rightarrow$  B always lies. Contradiction to "B tells truth". Case 1) is impossible.

In 2),

A tells truth  $\Rightarrow$  B always lies  $\Rightarrow$  A lies. Contradiction leads us to the conclusion, case 2) is impossible

In 3),

A lies  $\Rightarrow$  B tells the truth  $\Rightarrow$  A always tells the truth. We have a contradiction. Case 3) is impossible

In 4),

A lies  $\Rightarrow$  B tells the truth  $\Rightarrow$  Contradiction to "B lies". Case 4) is impossible.

Conclusion: Prof. Lai lies.

HW Problem. A certain country is inhabited only by people who either always tell the truth or always tell lies, and who will respond to questions only with a "yes" or a "no". A tourist comes to a fork in the road, where one branch leads to the capital and the other does not. There is no sign indicating which branch to take, but there is an inhabitant, Mr. Z, standing at the fork. What single question should the tourist ask him to determine which branch to take?

## The Principle of Mathematical Induction

Let me remind you that

$$1+2+3+\dots+(n-2)+(n-1)+n = \frac{n(n+1)}{2} \text{ for all } n=1,2,3,\dots$$

This formula can be proved by noticing that

$$1+n=n+1, \quad 2+(n-1)=n+1, \quad 3+(n-2)=n+1, \quad \dots$$

However, as in the following example

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1} \text{ for all } n=1,2,3,\dots$$

Sometimes it is difficult to get some formula. We cannot justify the formula by verifying that it holds for each individual value of  $n$  since there are infinitely many positive integers. Fortunately there is a formal scheme for proving statements are true for all positive integers; this scheme is called the principle of mathematical induction.

The principle of Mathematical Induction. Let  $S(n)$  be a proposition involving the integer  $n$ . Suppose that for some fixed integer  $n_0$

- (1)  $S(n_0)$  is true and
- (2) whenever  $k$  is an integer such that  $k \geq n_0$  and  $S(k)$  is true, then  $S(k+1)$  is true.

Then  $S(n)$  is true for all integers  $n \geq n_0$ .

A proof by mathematical induction consists of two parts. Part (1) establishes a base for the induction by proving that some statement  $S(n_0)$  is true.

Part (2), called the inductive step, proves that if any proposition  $S(k)$  is true, then so is the next proposition  $S(k+1)$ .

Example 1. Prove that  $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}, \quad \forall n=1,2,3,\dots$

Proof: We will prove this equality with help of the principle of Mathematical induction.

Step 1: Base of M.I:  $n_0=1$ ,  $\frac{1}{1 \cdot 2} = \frac{1}{2}$ . Therefore, equality holds for  $n_0=1$ .

Step 2 - Inductive step:  $k \rightsquigarrow k+1$

Assume  $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}$  holds for some  $k \geq 1$

Then

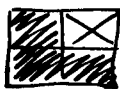
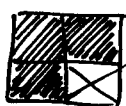
$$\begin{aligned} \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \\ &= \frac{(k+2)k+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}, \end{aligned}$$

i.e.  $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$ , or the same  
the equality holds for  $k+1$ .

Then, by the principle of Mathematical Induction, the equality holds for all  $n \geq 1$ .  $\square$

Example 2. Show that, for any integer  $n$ , if any one square is removed from a  $2^n \times 2^n$  checkerboard (one having  $2^n$  squares in each row and column), then the remaining squares can be covered with L-shaped pieces that cover three squares.

3	3	4	4
3	1	1	4
2	X	1	5
2	2	5	5



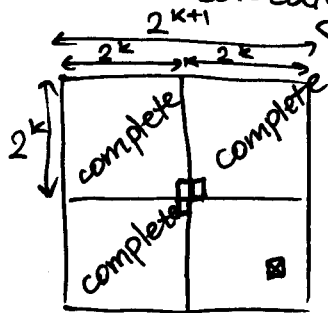
Case  $n=1$ .

Solution: Picture shows that every  $2^1 \times 2^1$  checkerboard with one square removed can be covered with a single L-shaped piece. Hence the result is true for  $n=1$ . This is our base  $n=1$  (Step 1).

Step 2:  $k \rightsquigarrow k+1$ .

Now assume that the result is true for some positive integer  $k$ , that is, every  $2^k \times 2^k$  checkerboard with one square removed can be covered by L-shaped pieces.

Take  $2^{k+1} \times 2^{k+1}$  checkerboard and divide it in half both horizontally and vertically



One of these  $2^k \times 2^k$  checkerboards has a square removed, and the other three are complete.

From each of the complete  $2^k \times 2^k$  checkerboards remove the square that touches the center of the original  $2^{k+1} \times 2^{k+1}$  checkerboard. By the induction hypothesis, all of four of  $2^k \times 2^k$  checkerboard can be covered with L-shaped pieces. Hence, with



one more L-shaped piece to cover the three squares touching the center of  $2^{k+1} \times 2^{k+1}$  checkerboard, we can cover with L-shaped pieces the original  $2^{k+1} \times 2^{k+1}$  checkerboard with one square removed. This proves the result for  $k+1$ .  
By the principle of M.I., the result is true for all  $n \geq 1$ .

Example 3. Prove that  $n! > 2^n$  if  $n \geq 4$ .

Solution: ① If  $n=4$ ,  $4! = 24$ ,  $2^4 = 16$ . So statement holds

② Suppose  $k! > 2^k$  for some  $k \geq 4$ . Then

$$(k+1)! = k!(k+1) > 2^k(k+1) \geq 5 \cdot 2^k > 2 \cdot 2^k = 2^{k+1}$$

This is the required inequality for  $k+1$ .

Thus, by the induction principle, the statement holds for all  $n \geq 4$   $\square$

Example 4. Determine what's wrong with the given induction arguments.

We will prove that in any set of  $n$  persons, all people have the same age.

Clearly, all people in a set of 1 person have the same age. So the statement is true if  $n=1$ . This is our base

Now suppose that in any set of  $k$  people all persons have the same age. Let  $S = \{x_1, x_2, \dots, x_{k+1}\}$  be the set of  $(k+1)$  people

Then by induction hypothesis all people in each of the sets  $\{x_1, x_2, \dots, x_k\}$  and  $\{x_2, x_3, \dots, x_{k+1}\}$  have the same age.

But then  $x_1, \dots, x_k$  all have the same age, and  $x_2, x_3, \dots, x_{k+1}$  have the same age. It follows that  $x_1, x_2, \dots, x_{k+1}$  all have the same age. This proves the inductive step.

By the principle of M.I., <sup>for</sup> any positive integer  $n$ , all people in any set of  $n$  persons have the same age.

HW Problem. Prove that

(a)  $1(1!) + 2(2!) + \dots + n(n!) = (n+1)! - 1$ , for all  $n \in \mathbb{N}$

(b)  $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} > \sqrt{n}$  for any positive integer  $\sqrt{n} > 1$