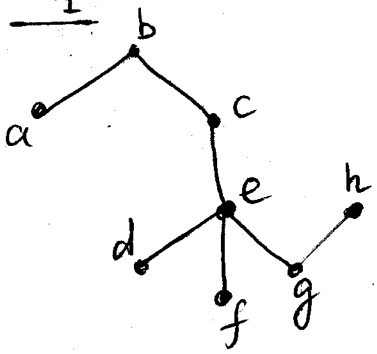
Trees.

We now will consider a special type of graph called a tree.

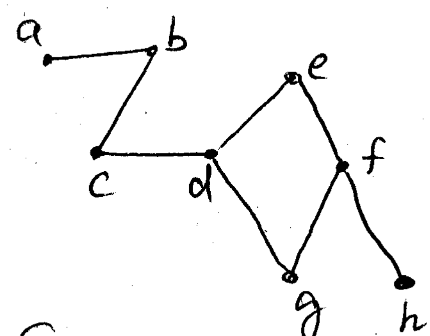
<u>Definition</u>. Let $G=(V,E)$ be a loop-free undirected graph. The graph G is called a tree if G is connected and contains no cycles.

G_1:



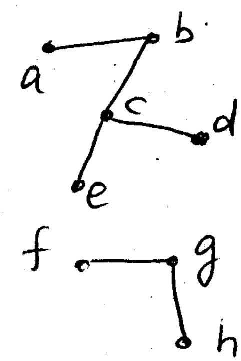
G_1 is a tree

G_2:



G_2 is not a tree

G_3:



G_3 is not a tree

<u>Properties of a tree:</u>

Let $T=(V,E)$ be a tree. Then

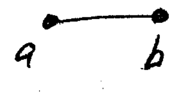
1) For any two vertices a and b there is a unique path from a to b
2) $|V| = |E|+1$
3) If $|V| \geq 2$ then T has at least two vertices of degree 1.

Proof of Properties:

1) Assume contrary, i.e. T has two or more paths for some pair of vertices, say a and b. Let P_1 and P_2 be two a-b paths. Then $P_1 \to P_2$ is a closed a-a walk. It implies that $\exists$ a circuit and hence a cycle in T, i.e.

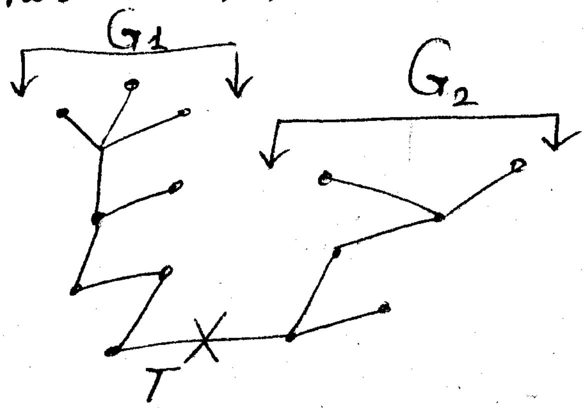
T is not a tree. This contradiction means that for any two vertices a, b there is a unique path from a to b.

2) Let $|E| = 1$, then T is the following graph



and we have $|V|=2$, $|E|=1$, $|V|=|E|+1$. Assume $|V|=|E|+1$ for any tree with k or less edges. Let us prove that $|V|=|E|+1$ for a tree with $(k+1)$ edges.

Let $T=(V,E)$ be a tree with $(k+1)$ edges.



Let us remove one edge from T. We obtain two subtrees $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$, where $|V_1|+|V_2|=|V|$ and $|E_1|+|E_2|+1=|E|$.

It follows by the induction hypothesis that $|E_1|+1=|V_1|$ and $|E_2|+1=|V_2|$. Hence,

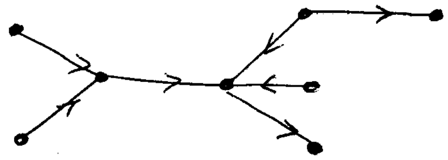
$|V|=|V_1|+|V_2|=|E_1|+1+|E_2|+1=\left(|E_1|+|E_2|+1\right)+1=|E|+1$.

It proves the property 2).

3) Let $|V|=n\geq 2$. By Property 2) $|E|=n-1$. Hence $2|E|=\sum_{v\in V}\deg(v)=2(n-1)$. Since T is connected then $\deg(v)\geq 1$, $\forall v\in V$. Assume that T has fewer than two vertices of degree 1, then either $\deg(v)\geq 2$, $\forall v\in V$, or $\deg(v_0)=1$ for only one vertex $v_0\in V$. In the first case $\sum_{v\in V}\deg(v)\geq 2n>2(n-1)$. In the second case, $\sum_{v\in V}\deg(v)\geq 1+2(n-1)>2(n-1)$. In both casses we have a contradiction to $\sum_{v\in V}\deg(v)=2(n-1)$. It means that T has two or more vertices of degree 1.

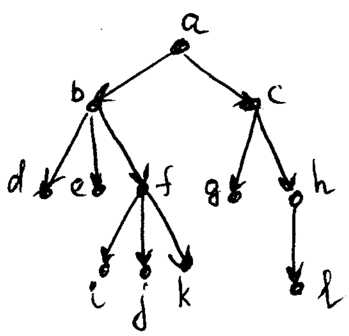
Rooted trees.

<u>Definition.</u> A directed graph is said to be a <u>directed tree</u> if it becomes a tree when the directions of the edges are ignored.



G is a directed graph that is a directed tree.

<u>Definition.</u> A directed tree is called a <u>rooted tree</u> if there is exactly one vertex that does not have incoming edges and all other vertices have exactly one incoming edge. A vertex with no incoming edge is called a <u>root</u>.



$T=(V,E)$ is a directed tree that is a rooted tree

- A vertex with no outcoming edge is called a <u>leaf</u>.

a is a root
d, e, i, j, k, g, l are leaves.

- A vertex who has outcoming edges is called a <u>branch node</u>.

b, f, c, h are branch nodes.

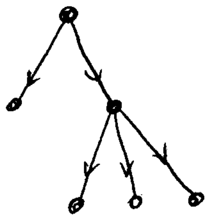
- Let v be a branch node in a rooted tree. A vertex w is called a <u>son</u> of v if there is an edge from v to w. Also, v is called a <u>father</u> of w

i, j, k are sons of f; c and b are sons of a.
b is a father of d, e, f; h is a father of l.

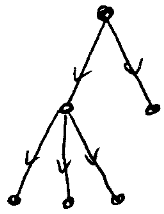
- Two vertices are said to be <u>brothers</u> if they are sons of the same vertex.

g and h are brothers; i, j and k are brothers

Consider the rooted tree which is the family tree of a man who has two sons, with the older son having no children and the younger son having three.

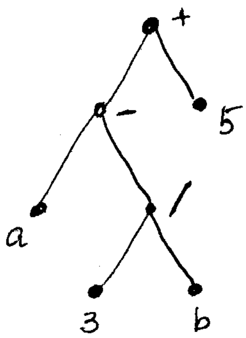


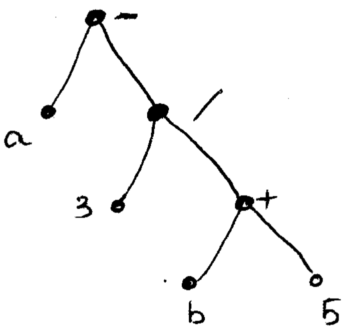
or



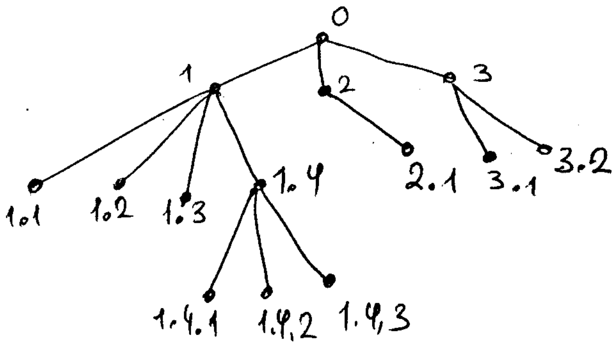
Ex. Find the rooted trees for the algebraic expressions

(i) $(a - \frac{3}{b}) + 5$ (ii) $a - (3/(b+5))$





Usually in a rooted tree, vertices are ordered from left to right. Such trees are called ordered rooted trees.



4-ary tree

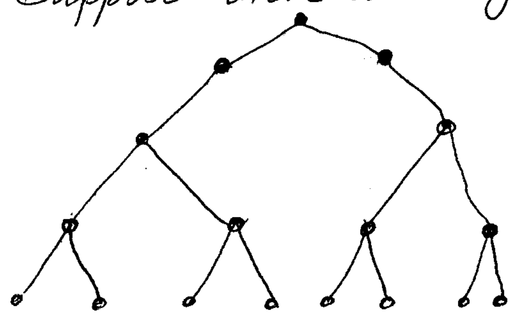
Definition. An ordered tree in which every branch node has at most m sons is called m-ary tree.
An m-ary tree is said to be regular if every one of its branch nodes has exactly m sons.

Example 1. Show that a regular binary tree has an odd number of vertices.

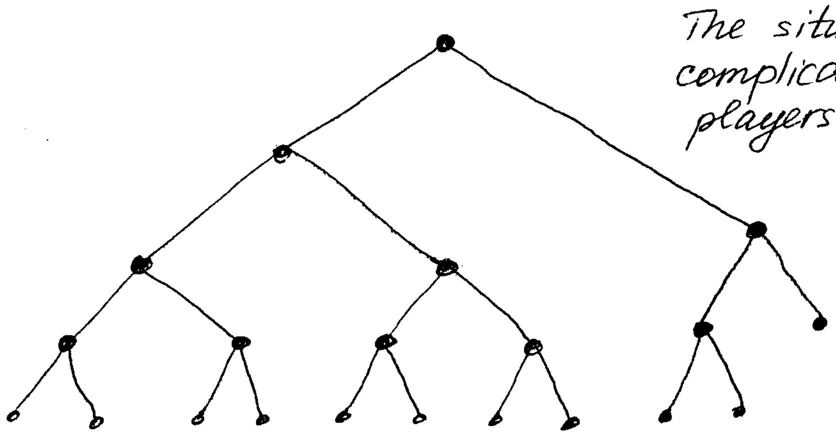
Solution. We have a root, intermidiate nodes and leaves. Since deg(root) = 2 = even, deg(int.) = 3, deg(leaf) = 1, then the number of interm. nodes and leaves must be even. Hence, if we add also a root, then the number of vertices becomes odd.

Path lengths in rooted trees.

Consider the problem of determining the number of games played in a single-elimination tennis tournament. Suppose there are eight players in the tournament.



There will be four games to be played in the first round, two games to be played in the second round and one game to be played in the final game. Totally, 7 games



The situation is more complicated if we have 11 players.

In this case there will be five games in the first round, three games in the second round and one game in the third round and one game in the fourth round. Totally 10 games.

We see that any regular binary tree can be viewed as the schedule of a single-elimination tournament. Let i be the number of branch nodes, or the same the number of games, let t be the number of players, or the number of leaves in a regular binary tree. Then

$$i = t - 1$$

(since each game eliminates a player, and at the end of the tournament, all players but the champion are eliminated).

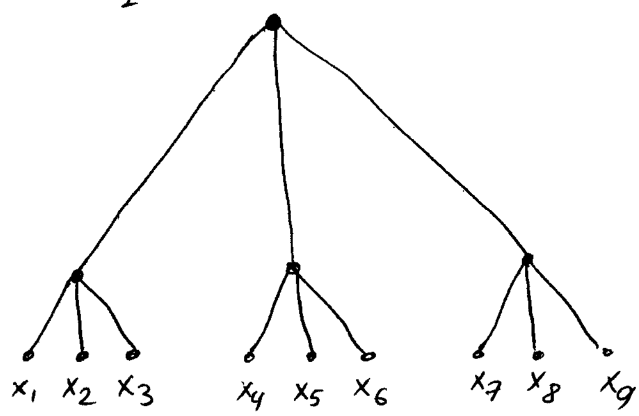
The result can be extended to the case of regular m-ary trees.

Let i be the number of branch nodes, t be the number of leaves in a regular m-ary tree. Then

$$(m-1)i = t - 1.$$

Example 2. Let us consider a hypothetical computer that has an instruction which computes the sum of three numbers. Suppose we want to find the sum of nine numbers, $x_1, x_2, \dots, x_9$. Consider a regular ternary tree with nine leaves.

T_1:



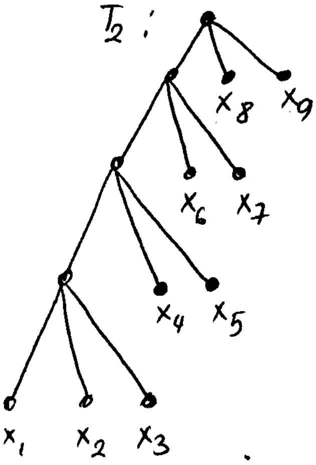
We have

$$(3-1)i = 9-1, \quad i = 4,$$

i.e. the addition instruction will always be executed four times.

However, we can consider another ternary tree with 9 leaves.

T_2:



Definition. The path length of a vertex in a rooted tree is defined to be the number of edges in the path from the root to the vertex.

For example, the path length of x_5 in T_2 is 3; the path length of x_1 in T_2 is 4.

Definition. The height of a tree is maximum of path lengths in the tree.

For example, the height of T_1 is 2; the height of T_2 is 4.

Remark: m-ary tree of height h has $\leq m^h$ leaves.

Example 3. A regular ternary (or 3-ary) tree $T=(V,E)$ has 34 internal vertices (including a root). How many edges does T have? How many leaves?

Solution: We have $(3-1)\cdot 34 = t-1 \Rightarrow t = 1+68 = 69$, i.e. T has 69 leaves. Hence, T has $69+34 = 103$ vertices and therefore 102 edges.

Groups.

Definition. Let G be a nonempty set and $\circ$ be a binary operation on G, i.e. $\circ: G\times G \to G$.
Set G is called a group if the following conditions are satisfied

1) $\forall a, b \in G,\quad a\circ b \in G$.
2) $\forall a, b, c \in G,\quad a\circ(b\circ c) = (a\circ b)\circ c$ (The Associative Property)
3) $\exists\, e \in G$ s.t. $a\circ e = e\circ a = a \quad \forall a \in G$ (The existence of an Identity)
4) $\forall a \in G\ \exists\, b \in G$ s.t. $a\circ b = b\circ a = e$ (Existence of Inverses)

Definition. Let G be a group. If $a\circ b = b\circ a \quad \forall a, b \in G$, then G is called a commutative group.

Example.

① $G = \{-1, 1\}$, $\circ$ is usual multiplication

(i) $1\circ 1 = 1 \in G$, $1\circ(-1) = -1 \in G$, $(-1)\circ 1 = -1 \in G$, $(-1)\circ(-1) = 1 \in G$

(ii) $a\circ(b\circ c) = a(bc) = (ab)c = (a\circ b)\circ c$, $\forall a, b, c \in G$

(iii) $\exists\, e = 1$ s.t. $1\circ e = 1$, $(-1)\circ e = -1$

(iv) the inverse of 1 is 1, the inverse of -1 is -1.
Hence G is a group

② $G = \{-1, 0, 1\}$, $\circ$ is usual addition.

(i) $1\circ 1 = 1+1 = 2 \notin G$. Hence G is not a group

③ $G = \{\frac{a}{2^n} ; a, n \in \mathbb{Z}, n \geq 0\}$, $\circ$ is usual addition.

(i) $\frac{a}{2^n}\circ\frac{b}{2^m} = \frac{a}{2^n} + \frac{b}{2^m} = \frac{2^m a + 2^n b}{2^{nm}} \in G$

(ii) $\frac{a}{2^n}\circ\left(\frac{b}{2^m}\circ\frac{c}{2^k}\right) = \frac{a}{2^n} + \left(\frac{b}{2^m} + \frac{c}{2^k}\right) = \left(\frac{a}{2^n} + \frac{b}{2^m}\right) + \frac{c}{2^k} = \left(\frac{a}{2^n}\circ\frac{b}{2^m}\right)\circ\frac{c}{2^k}$.

(iii) $e = \frac{0}{2^0} = 0$. Then $\frac{a}{2^n}\circ\frac{0}{2^0} = \frac{a}{2^n} = 0\circ\frac{a}{2^n}$.

(iv) $\forall\ \frac{a}{2^n}\ \exists\ -\frac{a}{2^n}$ s.t. $\frac{a}{2^n}\circ\left(-\frac{a}{2^n}\right) = e = 0$.

Theorem For every group G

(a) the identity of G is unique;
(b) the inverse of each element of G is unique;
(c) $ab = ac \Rightarrow b = c, \ \forall a, b, c \in G$;

(d) $ba = ca \Rightarrow b = c, \ \forall a, b, c \in G$.
(e) $(a \circ b)^{-1} = b^{-1} \circ a^{-1}$.

Proof:

(a) Assume that there are two identities, say e_1 and e_2, then (i) $a \circ e_1 = a \quad \forall a \in G$. Hence $e_2 \circ e_1 = e_2$
(ii) $b \circ e_2 = b \quad \forall b \in G$. Hence $e_1 \circ e_2 = e_1$

Since $a \circ e = e \circ a = a \quad \forall a \in G$ then $e_2 \circ e_1 = e_1 \circ e_2$ and therefore $e_1 = e_2$.

(b) Assume that some $a \in G$ has two inverses, say b_1 and b_2. Then

$$a \circ b_1 = e \Rightarrow b_2 \circ (a \circ b_1) = b_2 \circ e = b_2$$
$$\Downarrow$$
$$(b_2 \circ a) \circ b_1 = b_2$$
$$\Downarrow$$
$$e \circ b_1 = b_2$$
$$\Downarrow$$
$$b_1 = b_2$$

(c) $a \circ b = a \circ c \Rightarrow a^{-1} \circ (a \circ b) = a^{-1} \circ (a \circ c)$, where a^{-1} is notation for the inverse of $a \Rightarrow$
$(a^{-1} \circ a) \circ b = (a^{-1} \circ a) \circ c \Rightarrow e \circ b = e \circ c \Rightarrow b = c$.

HW (d) $ba = ca \Rightarrow b = c$ ■
(e) $(a \circ b) \circ (b^{-1} \circ a^{-1}) = (a \circ (b \circ b^{-1}) \circ a^{-1}) = e \Rightarrow (a \circ b)^{-1} = b^{-1} \circ a^{-1}$.

Further, the inverse of a we will denote by a^{-1}.

Problem 1. Prove that G is commutative iff for all $a, b \in G$, $(a \circ b)^{-1} = a^{-1} \circ b^{-1}$.

Proof.

(i) Let G be commutative. Then
$$(a \circ b)^{-1} = b^{-1} \circ a^{-1} = a^{-1} \circ b^{-1}.$$
(ii) Let $(a \circ b)^{-1} = a^{-1} \circ b^{-1}, \ \forall a, b \in G$. Let us show G is commutative.
We have, $(a \circ b)^{-1} = b^{-1} \circ a^{-1} \Rightarrow ((a \circ b)^{-1})^{-1} = (a^{-1})^{-1} \circ (b^{-1})^{-1} = a \circ b$
$(a \circ b)^{-1} = a^{-1} \circ b^{-1} \Rightarrow ((a \circ b)^{-1})^{-1} = (b^{-1})^{-1} \circ (a^{-1})^{-1} = b \circ a$ ■