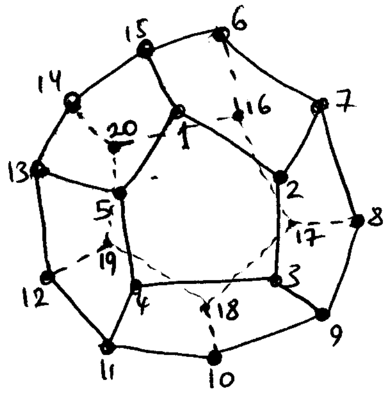
Hamiltonian cycles (Continuation)

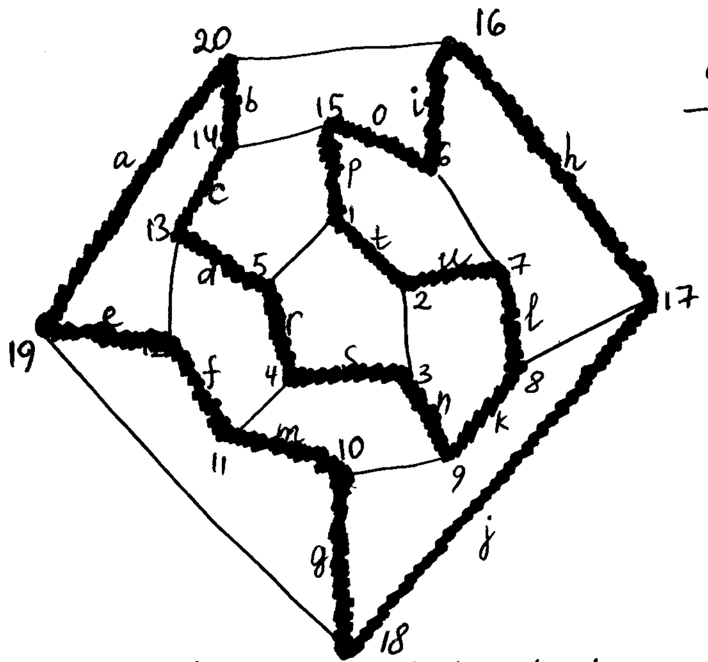
Hamiltonian invented a game which made use of a wooden regular dodecahedron, that is, a solid with 12 congruent faces, each of which is a regular pentagon.



Dodecahedron

The vertices of the dodecahedron were labeled with the names of 20 cities of the world, and the aim of the game was to find a route "around the world", along the edges of the solid, which passed through each city exactly once and led back to the city where the tour started.

Imagine that the pentagon on which the dodecahedron sits is stretched so that the solid collapses until it is flat. The result is a graph on a plane.



Stretched dodecahedron

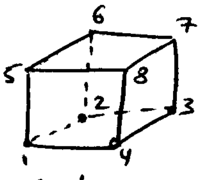
Hamiltonian's tour "around the world" is possible if and only if the graph contains what we now call a Hamiltonian cycle. The graph in fact contains a Hamiltonian cycle. A Hamiltonian cycle is marked with the heavy lines.

To construct such Hamiltonian cycle H we use property of cycles. We have inner vertices 1, 2, 3, 4, 5, middle vertices 6, 7, 8, 9, 10, 11, 12, 13, 14, 15 and outer vertices 16, 17, 18, 19, 20.
Cycle must connect outer vertices with middle vertices.
Let $\{20,19\}\in H$, $\{20,14\}\in H \Rightarrow \{14,15\}\notin H \Rightarrow \{14,13\}\in H$. Assume $\{13,5\}\in H$ $\Rightarrow \{13,12\}\notin H \Rightarrow \{19,12\}\in H$, $\{12,11\}\in H$. Also, $\{20,16\}\notin H \Rightarrow \{16,17\}\in H$, $\{16,6\}\in H$. Since $\{19,18\}\notin H$, then $\{18,10\}\in H$, $\{18,17\}\in H$. Hence $\{17,8\}\notin H$ $\Rightarrow \{8,9\}\in H$, $\{7,8\}\in H$. Assume $\{10,11\}\in H$, ~~then~~ $\{10,9\}\notin H \Rightarrow \{3,9\}\in H$
We have, $\{14,15\}\notin H \Rightarrow \{15,6\}\in H$, $\{15,1\}\in H$. Since $\{11,4\}\notin H \Rightarrow \{5,4\}\in H$, $\{4,3\}\in H$.

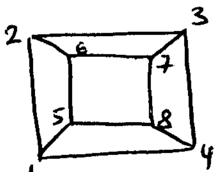
Since $\{1,5\}\notin H$ then $\{1,2\}\in H$. Also, $\{2,3\}\notin H$. Hence $\{2,7\}\in H$.

Problem 1. Draw a picture of a cube. By imagining that the bottom square is stretched until it is flat, draw a graph of the flattened cube. Is this graph Hamiltonian, i.e. does this graph have a Hamiltonian cycle? If so, draw a Hamiltonian cycle. If not, explain why not.

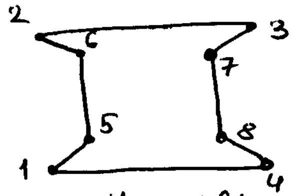
Solution.



cube



stretched cube



a Hamiltonian cycle.

Problem 2. A connected graph G has 11 vertices and 53 edges. Show that G is Hamiltonian, i.e. G has a Hamiltonian cycle. Show that G does not have an Euler circuit.

Solution:

① Let $G=(V,E)$, $|V|=11$, $|E|=53$.
We have, $\deg(x)\le 10$, $\forall x\in V$ (we consider a graph, not multigraph).
From $\sum_{x\in V}\deg(x)=2|E|$, i.e. $\sum_{x\in V}\deg(x)=106$, it follows $\deg(x)>5$ for all $x\in V$. Indeed, if there is one vertex with $\deg(x)\le 5$ then $\sum_{x\in V}\deg(x)\le 10\cdot 10+5=105$. Hence, all vertices have degrees >5. Hence, $\deg(x)+\deg(y)\ge 12$ $\forall x\in V, y\in V$. By the Theorem proved last time, G has a Hamiltonian cycle.

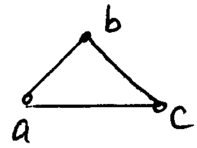
② Let us show that $G=(V,E)$, $|V|=11$, $|E|=53$ does not have an Euler circuit.
Claim: G has 7 or more vertices of degree 10.
Indeed, if not, then $\sum_{x\in V}\deg x\le 6\cdot 10+5\cdot 9=105<106$.
Take seven vertices of degree 10 away and consider the other four vertices, say x_1, x_2, x_3, x_4. We have $\deg x_1+\deg x_2+\deg x_3+\deg x_4=36$.
It is easy to see that $\deg x_i\ge 6$ for all $i=1,2,3,4$. Indeed, if $\exists x_j$ with $\deg x_j\le 5$ then $\sum_{i=1}^{4}\deg x_i\le 10\times 3+5=35$. Hence $6\le\deg x_i\le 10$, $\forall i=1,2,3,4$. We have the following possibilities
$\deg(x_1)=6$, $\deg(x_2,x_3,x_4)=10 \Rightarrow$ 1) $\deg(x)=10$ $\forall x\in V$ except y and $\deg y=6$
$\deg(x_1)=8$, $\deg(x_2,x_3)=10$, $x_4=8\Rightarrow$ 2) $\deg(x)=10$ $\forall x\in V$ except y,z and $\deg y=\deg z=8$

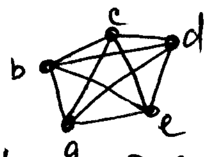
If $\deg(x_1) = 7$ or 9 then G does not have an Euler circuit.
Case 1) does not hold, since if $\deg(x)=10$, $\forall x \in V$ except $y \in V$, then $\deg(y) = 10$ also.
Case 2) does not hold as well. Indeed, if $\deg(x)=10$, $\forall x \in V$ except y and z, Then $\deg(y) \geq 9$ and $\deg(z) \geq 8$. Hence $\deg(x_1) = 7$ or 9 and then G does not possess an Euler circuit.

Problem 3. Show a graph that
(a) has both an Euler circuit and a Hamiltonian cycle;
(b) has no Euler circuit but has a Hamiltonian circuit;
(c) has an Euler circuit but has no a Hamiltohian cycle;
(d) has neither an Euler circuit, nor a Hamiltonian cycle.

Solution:

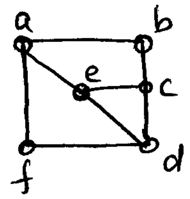
(a)





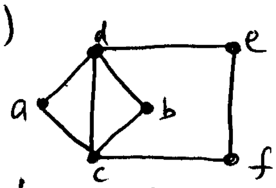
has an Euler circuit and a Hamiltonian cycle

(b)



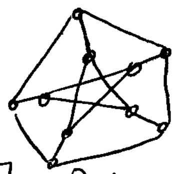
has a Hamiltonian cycle $\{a,b\}, \{b,c\}, \{c,e\}, \{e,d\}, \{d,f\}, \{f,a\}$ and does not have an Euler circuit ($\deg(a)=3$)

(c)



has an Euler circuit (degrees of all vertices are even numbers) and does not have an Hamiltonian cycle ($\{a,d\}, \{d,b\}, \{b,c\}, \{c,a\}$ is a cycle that will be in another cycle).

(d)



The Petersen graph does not have an Euler circuit ($\deg(x)=3$) and it does not have a Hamiltonian cycle.