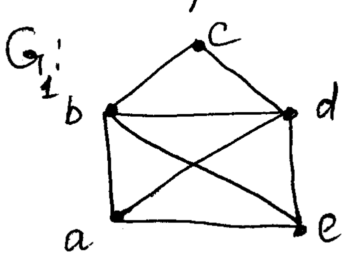
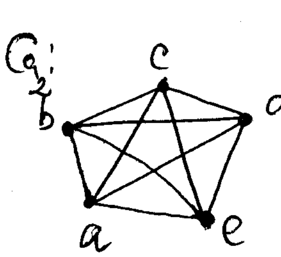
Euler Trails and Circuits.

Definition. Let $G=(V,E)$ be an undirected graph or multigraph with no isolated vertices. Then G is said to have an Euler circuit if there is a circuit in G that passes over every edge of the graph exactly once. If there is an open trail from a to b in G and this trail traverses each edge exactly once, the trail is called an Euler trail.

Example.



$\{a,b\},\{b,c\},\{c,d\},\{d,b\},\{b,e\},\{e,d\},\{d,a\},\{a,e\}$ is an Euler trail in G_1.



$\{a,c\},\{c,e\},\{e,b\},\{b,d\},\{d,a\},\{a,b\},\{b,c\},\{c,d\},\{d,e\},\{e,a\}$ is an Euler circuit in G_2.

Theorem 1. Let $G=(V,E)$ be an undirected graph or multigraph with no isolated vertices. Then G has an Euler circuit if and only if G is connected and every vertex in G has even degree (without proof).

Theorem 2. An undirected graph possesses an Euler trail if it is connected and has either zero or two vertices of odd degree (without proof).

Example. G_1 possesses an Euler trail ($\deg(a)=\deg(e)=3$, $\deg(c)=2$, $\deg(b)=\deg(d)=4$) and does not possess an Euler circuit.

Euler Circuit Algorithm.

This algorithm finds an Euler circuit for a connected multigraph G in which every vertex has even positive degree.

<u>Step 1.</u> Select a vertex v and an edge $\{v, u\}$.

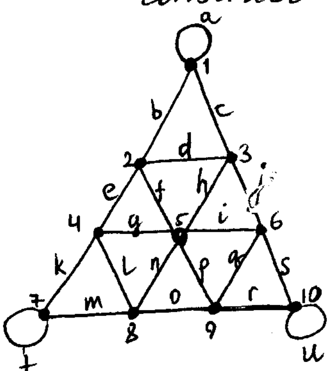
<u>Step 2.</u> If the other vertex on the last chosen edge is v, then go to Step 3. Otherwise choose an unused edge starting at u and repeat step 2.

<u>Step 3.</u> If all of the edges have been used, then stop (an Euler circuit has been constructed). Otherwise choose an unused edge on a vertex already visited, give this previously visited vertex a temporary name A.

<u>Step 4.</u> (get a path from a to a). If the other vertex on the last chosen edge is not a, then choose an unused edge on this other vertex and repeat Step 4.

<u>Step 5.</u> (join paths together) Insert these newly chosen edges at the vertex a. Go to Step 3.

<u>Problem 1.</u> Determine if the multigraph has an Euler circuit. If yes, construct it.



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Solution:

$deg(1) = deg(2) = deg(3) = deg(4) = deg(6) =$
$= deg(7) = deg(10) = deg(8) = deg(9) = 4$
$deg(5) = 6$

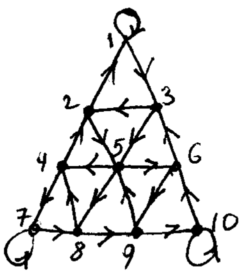
Therefore, G has an Euler circuit

Take 1, consider path
1 – 1: c, d, b
1 – 1: a
→ a, c, d, b
2 – 2: f, g, e
→ a, c, d, f, g, e, b
3 – 3: h, i, j

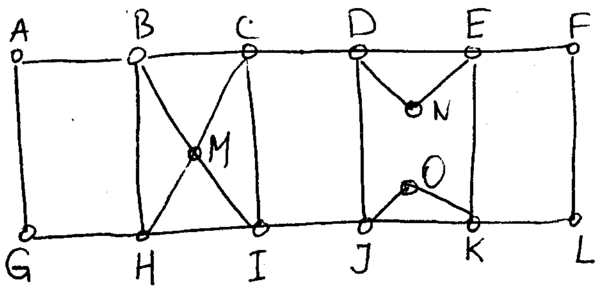
a, c, h, i, j, d, f, g, e, b
4 – 4: k, t, m, l
→ a, c, h, i, j, d, f, g, k, t, m, l, e, b
5 – 5: n, o, r, u, s, q, p

An Euler circuit in G is

a, c, h, n, o, r, u, s, q, p, i, j, d, f, g, k, t, m, l, e, b



Problem 2. A power company's wires in a certain region follow the routes indicated by the following graph



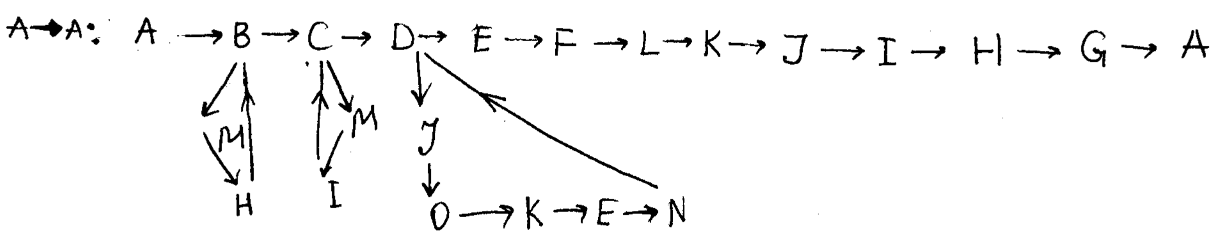
The vertices represent poles and the edges wires. After a severe storm, all the wires and poles must be inspected. Show that there is a round trip beginning at A, which allows a person to inspect each wire once. Find such a trip.

Solution:

deg(A) = deg(G) = deg(F) = deg(L) = 2

deg(B) = deg(C) = deg(H) = deg(I) = deg(D) = deg(E) = deg(J) = deg(K) = 4

Therefore, such graph has an Euler circuit. It means that a person can find a round trip from A to A and inspect each wire once.



An Euler circuit is the following (a trip for an inspection of all wires)

A → B → M → H → B → C → M → I → C → D → J → O → K → E → N → D → E →

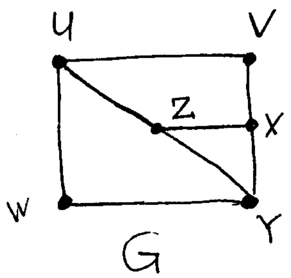
F → L → K → J → I → H → G → A.

Hamiltonian Cycles and Paths

<u>Definition.</u> In a graph a Hamiltonian path is a path that contains each vertex once and only once.
Hamiltonian cycle is a cycle that includes each vertex.

These are named after Sir William Rowan Hamilton, who developed a puzzle where the answer required the construction of this kind of cycle.

Example.



$\{U,V\}, \{V,X\}, \{X,Z\}, \{Z,Y\}, \{Y,W\}, \{W,U\}$ is a Hamiltonian cycle

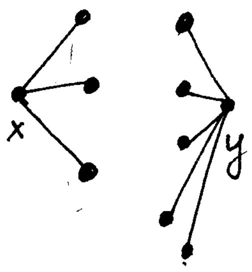
(Suppose, the graph describes a system of airlines routes between vertices (towns).
The vertex U is the home base for a salesman who must periodically visit all of the other cities. To be economical the salesperson wants a path that starts at U, ends at U, and visit each of the other vertices exactly once.

<u>Remark.</u> There exists an easy criteria to determine if there is an Euler circuit or an Euler path. There is a straightforward algorithm to use in constructing an Euler circuit. Unfortunately, the same does not hold for a Hamiltonian cycle and a Hamiltonian path. In general, it is very difficult to find a <u>Hamiltonian cycle</u> for a graph. There are, however, some conditions that guarantee the existence of a Hamiltonian cycle in a graph.

<u>Theorem.</u> *Let G be a (linear) graph with n vertices. If the sum of the degrees for each pair of vertices in G is $n-1$ or larger, then there exists a hamiltonian path in G.*

<u>Proof.</u> Assume that $G=(V,E)$ with $|V|=n$ and $\forall v,w\in V,\ v\neq w,\quad deg(v)+deg(w)\geq n-1$. Let us show that then G is connected.
Take any $x,y\in V$. If $\exists \{x,y\}\in E$ then there is a path from x to y. Assume that $\nexists\{x,y\}\in E$. Let S_1 be the set of vertices adjacent to x and S_2 be the set of vertices adjacent to y. Then $x,y\notin S_1\cup S_2$
Hence, $|E|\geq |S_1\cup S_2|+2=|S_1|+|S_2|-|S_1\cap S_2|+2$

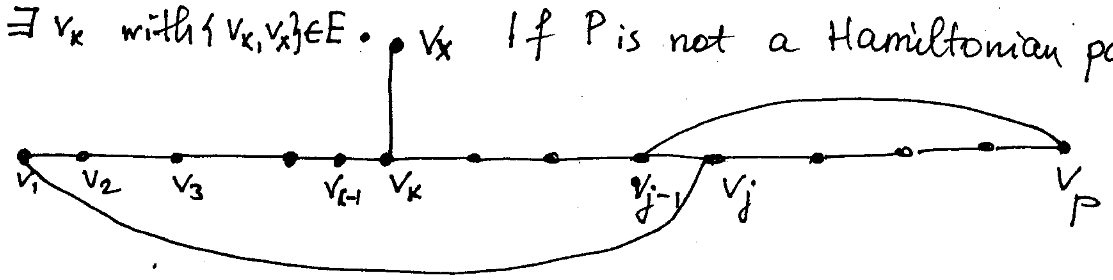
If $\exists\, z \in S_1 \cap S_2$, then $\{x, z\}, \{z, y\}$ will be a path from x to y.
Assume $S_1 \cap S_2 = \emptyset$, then $|S_1 \cap S_2| = 0$ and hence
$|V| \geq n-1+2 = n+1$. This contradiction means that $S_1 \cap S_2 \neq \emptyset$. Hence, there is a path from x to y. It follows that G is connected.



Let us show now that there is a hamiltonian path in G.

Let $P: v_1 \to v_2 \to \dots \to v_p$ be a path of $p-1$ edges, $p<n$.
If either v_1 or v_p is adjacent to a vertex that is not in the path P, we can immediately extend the path and include this vertex and obtain a path of p edges. Otherwise v_1 and v_p are adjacent to vertices in P. Then there is a circuit containing exactly the vertices $v_1, v_2, \dots, v_p$.
If v_1 is adjacent to v_p, then the circuit is $v_1 \to v_2 \to \dots \to v_p \to v_1$.
If no, assume v_1 is not adjacent to v_p. Let v_1 is adjacent to $v_{i_1}, v_{i_2}, \dots, v_{i_k}$, where $2 \leq i_j \leq p-1$.
If v_p is adjacent to one of $v_{i_1-1}, v_{i_2-1}, \dots, v_{i_k-1}$, say v_{j-1}, then the circuit is $v_1, v_2, \dots, v_{j-1}, v_p, v_{p-1}, \dots, v_j, v_1$.
If v_p is not adjacent to any one of $v_{i_1-1}, v_{i_2-1}, \dots, v_{i_k-1}$, then v_p is adjacent to $\leq n-k-2$ vertices. Consequently, $\deg(v_1) + \deg(v_p) = k + \deg(v_p) \leq n-2$. This contradiction means that v_p is adjacent to one of $v_{i_1-1}, \dots, v_{i_k-1}$.

Now we have a circuit containing all the vertices $v_1, v_2, \dots, v_p$. Take v_x that is not in the circuit and s.t. $\exists\, v_k$ with $\{v_k, v_x\} \in E$. If P is not a Hamiltonian path then

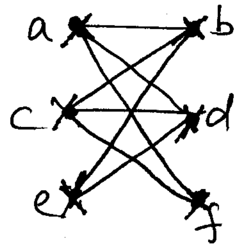


such v_x always exists (since G is connected). Then
$v_x, v_k, \dots, v_{j-1}, v_p, v_{p-1}, \dots, v_j, v_1, v_2, \dots, v_{k-1}$ is a path of length p
Again, $\exists$ a circuit connecting all these $p+1$ vertices.

If it is not a Hamiltonian cycle, we repeat adding another element (vertex) v_y and so on. ■

Corollary. Suppose G is a graph with n vertices, where $n > 2$. If each vertex has degre $\geq \frac{n}{2}$, then G has a Haimiltonian cycle.

Ex. 1:

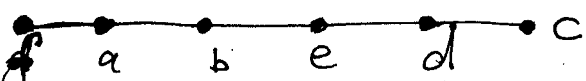


$G = (V, E)$

G has a Hamiltonian cycle, since

$$\deg(x) = 3 = \frac{6}{2} = \frac{|V|}{2} \quad \forall x \in V$$

P. $\{a,b\}, \{b,e\}, \{e,d\}, \{d,a\}$ of length 4



is a Hamiltonian path.

$f \to a \to b \to e \to d \to c \to f$ is a Hamiltonian cycle.

Ex 2. Consider the problem of scheduling seven examinations in seven days so that two examinations given by the same instructor are not scheduled on consecutive days. If no instructor gives more than four examinations, show that it is always possible to schedule the examinations.

Solution. Let G be a graph with seven vertices corresponding to the seven examinations. There is an edge between any two vertices which corresponds to two examinations given by different instructors. Since the degree of each vertex is ≥ 3, then $\forall x, y \in V$, $G = (V, E)$, $\deg(x) + \deg(y) \geq 6 = 7 - 1$, then G always contains a hamiltonian path, which corresponds to a suitable schedule for the seven examinations.

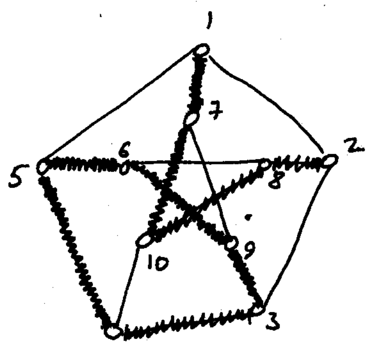
There are some properties of cycles which are helpful to conclude that a particular graph does not have a Hamiltonian cycle.

<u>Properties of cycles.</u> Suppose S is a cycle in a graph G.

1. For each vertex v of S, precisely two edges incident with v are in S; hence, if S is a Hamiltonian cycle of G and a vertex v in G has degree 2, then both edges incident with v must be part of S.
2. The only cycle contained in S is S itself.

<u>Proof of 2.</u> Suppose C is a cycle contained in S and $C \neq S$. Then $\exists$ vertex $y \in S$ but $y \notin S$. Let x be a vertex in in C. We have $x \in C$, $y \notin C$. Hence there is a path from x to y (in S). Thus S contains edge $\{u, v\}$ s.t. $u \in C$, $v \notin C$. So, S contains two edges of C that are incident with u together with the edge $\{u, v\}$. Totally, there are $\geq$ three edges incident with u which must be a part of S. This contradicts Property 1.

<u>Problem.</u> Consider the Petersen graph (a famous graph named after the Danish mathematician Julius Petersen 1839–1910). Show that the Petersen graph does not have a Hamiltonian ~~graf~~ cycle.



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<u>Solution.</u> Assume that G has an Hamiltonian cycle H. Then H must contain at least one of the five edges connecting the outer to the inner vertices. Since the graph is symmetric, WLOG, assume $\{1, 7\} \in H$. By Property 1, either $\{7, 10\} \in H$ or $\{7, 9\} \in H$. Again, by symmetry, assume $\{7, 10\} \in H$. Then $\{7, 9\} \notin H$. By property 1, $\{6, 9\} \in H$, $\{9, 3\} \in H$. Now, precizely one of $\{5, 6\}$, $\{6, 8\} \in H$. Consider case $\{5, 6\} \in H$. Another case, $\{6, 8\} \in H$, do as $\boxed{HW}$. Since $\{5, 6\} \in H$, $\{6, 9\} \in H$, then $\{6, 8\} \notin H$. Hence $\{8, 10\} \in H$, $\{2, 8\} \in H$. Wertex 10 already has 2 incident edges $\{7, 10\}$, $\{8, 10\}$ in H. Hence, $\{4, 10\} \notin H$. Then $\{5, 4\}, \{4, 3\} \in H$. Since H contains a cycle $\{5, 4\}, \{4, 3\}, \{3, 9\}, \{9, 6\}, \{6, 5\}$. Then H can not be Hamiltonian.