Sets

Any branch of science, like a foreign language, has its own terminology. Most people, at one time or another, have gone to a dictionary in search of a word only to discover that the definition uses another unfamiliar word. It indicates that a dictionary can be of no use unless there are some words which are so basic that we can understand them without definitions. Mathematics is the same way. There are a few basic terms which we accept without definitions.

Most of mathematics is based upon a single undefined concept of a <u>set</u>.

We think of a <u>set</u> as a collection of distinct objects such that given any object, we can tell whether that object is in the set or not. If the object x is in the set S we write $x \in S$, and if not we write $x \notin S$.

Examples: ① Let P be the set of all presidents of the United States. Then

$$\text{George Washington} \in P,$$
$$\text{Tony Blair} \notin P.$$

② If $\mathcal{U}$ is the set of all integers from 1 to 12. Then $5 \in \mathcal{U}$ but $15 \notin \mathcal{U}$.

If set has a finite number of objects, one way to define it is simply to list them all between curly braces. For example,

$$\mathcal{U}=\{1,2,3,4,5,6,7,8,9,10,11,12\}.$$

Another way to express a set is to enclose inside curly braces a description of a typical element of the set:

$$\mathcal{U}=\{x: x \text{ is an integer and } 0<x<13\}.$$

Remarks:

① Note that the elements in a set are not ordered in any fashion. Thus, $\{2,4,6\}$ and $\{6,2,4\}$ represent the same collection of elements.

② A set could contain no elements at all. The set that contains no element is called the empty set, and is denoted by $\varnothing$

Examples: ③ Let S be the set of all elephants that can fly. Then $S=\varnothing$.

④ Let $A=\{(x,y): x^2+y^2<0\}$. Evidently, $A=\varnothing$.

Definition. Two sets P and Q are said to be equal if P and Q contain the same elements.

Examples: ⑤ $P=\{t: t=r-s,\ r,s\in\{0,1,2\}\}$,
$Q=\{-2,-1,0,1,2\}$
Then $P=Q$.

⑥ Let $A=\{x: \sin x=0\}$, $B=\{\pi k: k \text{ is an integer}\}$.
Then $A=B$.

Definition. A set A is a subset of a set B, and we write $A\subset B$, if and only if every element of A is an element of B. If $A\subset B$, we say "A is contained in B".

Example: 7 $\{a,b\} \subset \{a,b,c\}$
$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

Combination of sets

Now we shall see how sets can be combined in various ways to yield new sets.

Definition. Let A and B be two sets.

The union of sets A and B, written $A \cup B$, is the set of elements in A or in B (or in both)

The intersection of sets A and B, written $A \cap B$, is the set of elements which belong to both A and B.

The difference of two sets A and B, written $A \setminus B$, is the set containing exactly those elements in A that are not in B.

We have

$$A \cup B = \{x : x \in A \text{ or } x \in B\},$$
$$A \cap B = \{x : x \in A \text{ and } x \in B\},$$
$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}.$$

Example: 8 $\{a,b\} \cup \{a,c\} = \{a,b,c\}$, $\{a,b,c\} \cap \varnothing = \varnothing$
$\{a,b\} \cap \{a,c\} = \{a\}$, $\{a,b,c\} \setminus \{a\} = \{b,c\}$
$\{a,b,c\} \cup \varnothing = \{a,b,c\}$,
$\{x : x^2 \geq 1\} \cap \{x : -\frac{1}{2} \leq x \leq \frac{1}{2}\} = \varnothing$.

In many applications all of the sets under consideration are subsets of some set $\mathcal{U}$. Such a set containing all of the elements of interest in a particular situation is called a universal set

Definition: Given a universal set $\mathcal{U}$ and a subset A of $\mathcal{U}$, the set $\mathcal{U} \setminus A$ is called the complement of A and is denoted by $\overline{A}$.

Example: 9 Let $A=\{1,2,4\}$, $B=\{2,4,6,8\}$. Let

$\mathcal{U}=\{1,2,3,4,5,6,7,8\}$ be the universal set.

Then $\overline{A}=\{3,5,6,7,8\}$ and $\overline{B}=\{1,3,5,7\}$.

Next theorem lists some elementary properties of set operations.

Theorem 1. Let $\mathcal{U}$ be a universal set. For any subsets A, B and C of $\mathcal{U}$, the following are true.

(a) $A\cup B=B\cup A$, $A\cap B=B\cap A$ (commutative laws)

(b) $(A\cup B)\cup C=A\cup(B\cup C)=: A\cup B\cup C$
$(A\cap B)\cap C=A\cap(B\cap C)=: A\cap B\cap C$ (associative laws)

(c) $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$
$A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$ (distributive laws)

(d) $\overline{\overline{A}}=A$

(e) $A\cup\overline{A}=\mathcal{U}$, $A\cap\overline{A}=\emptyset$

(f) $A\setminus B=A\cap\overline{B}$.

Proof of Theorem 1. We will prove only one equality from part (c) and part (f).

(c) To prove that $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$ we will show that each of the sets $A\cup(B\cap C)$ and $(A\cup B)\cap(A\cup C)$ is the subset of the other.

Let us prove that $\boxed{A\cup(B\cap C)\subset(A\cup B)\cap(A\cup C). \quad (*)}$

Take an arbitrary element $x\in A\cup(B\cap C)$. Then $x\in A$ or $x\in B\cap C$.

(i) If $x\in A$ then $x\in A\cup B$ and $x\in A\cup C$. Therefore, $x\in(A\cup B)\cap(A\cup C)$.

(ii) If $x\in B\cap C$ then $x\in B$ and $x\in C$. Therefore, $x\in A\cup B$ and $x\in A\cup C$. So, $x\in(A\cup B)\cap(A\cup C)$

In both cases ($x\in A$ or $x\in B\cap C$), x is an element of $(A\cup B)\cap(A\cup C)$. It proves $(*)$.

Let us prove that $\boxed{(A\cup B)\cap(A\cup C)\subset A\cup(B\cap C).\ (**)}$

Take an arbitrary element $x\in(A\cup B)\cap(A\cup C)$. Then $x\in A\cup B$ and $x\in A\cup C$.

(i) If $x\in A$ then $x\in A\cup(B\cap C)$.

(ii) if $x\in B\setminus A$ then $x\in C$. Therefore, $x\in B\cap C$. So, $x\in A\cup(B\cap C)$.

Therefore, if $x\in(A\cup B)\cap(A\cup C)$ then x is an element of $A\cup(B\cap C)$. It proves (**).

(f): To prove $A\setminus B=A\cap\overline{B}$ we will prove that (1) $A\setminus B\subset A\cap\overline{B}$ and (2) $A\cap\overline{B}\subset A\setminus B$.

Let us start with (1).

Take $x\in A\setminus B$. Then $x\in A$ and $x\notin B$. It is the same as $x\in A$ and $x\in\overline{B}$, or equivalently, $x\in A\cap\overline{B}$. It implies $A\setminus B\subset A\cap\overline{B}$.

Let us show (2).

Take $x\in A\cap\overline{B}$. It means $x\in A$ and $x\in\overline{B}$, or the same, $x\in A$ and $x\notin B$, or the same $x\in A\setminus B$. It implies $A\cap\overline{B}\subset A\setminus B$. $\square$

<u>Problem:</u> Prove that

$$A\subset B \quad \text{if and only if } A\cap\overline{B}=\emptyset.$$

<u>Solution:</u>

(1) Let $A\subset B$. Our aim is to show that $A\cap\overline{B}=\emptyset$. We have, by Theorem 1, (f), $A\cap\overline{B}=A\setminus B$. Clearly, $A\setminus B=\emptyset$ (since $A\subset B$). Therefore, $A\cap\overline{B}=\emptyset$.

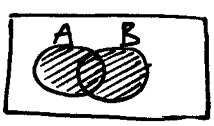
(2) Let $A\cap\overline{B}=\emptyset$. We have to show that $A\subset B$. Take $x\in A$. Then $x\notin\overline{B}$, i.e. $x\in\overline{\overline{B}}$. Since $B=\overline{\overline{B}}$

by Theorem 1, then $x \in B$. Therefore $A \subset B$. □

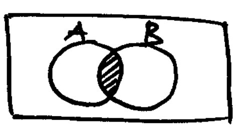
Venn diagrams.

Relationship among sets can be pictured in Venn diagrams, which are named after the English Logician John Venn (1834-1923). In a Venn diagram, the universal set is represented by a rectangular region and subsets of the universal set are usually represented by circular discs drawn within the rectangular region. Sets that are not known to be disjoint should be represented by overlapping circles.

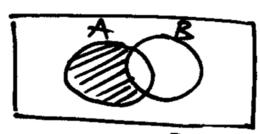
In each diagram the colored region depicts the set being represented.



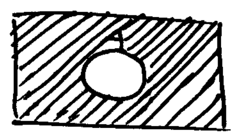
$A \cup B$



$A \cap B$

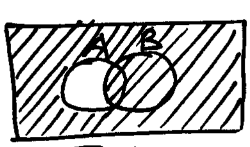


$A \setminus B$

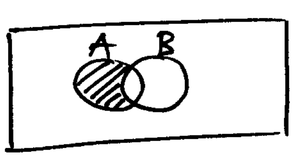


$\bar{A}$

More complicated sets can be constructed with help of Venn diagram.



$\bar{A} \cup B$



$\overline{\bar{A} \cup B}$

From the last picture we see that $\overline{\bar{A} \cup B} = A \setminus B$.

De Morgan's Laws

Theorem 2. For any subsets A and B of a universal set $\mathcal{U}$, the following are true.

(a) $\overline{A \cup B} = \bar{A} \cap \bar{B}$

(b) $\overline{A \cap B} = \bar{A} \cup \bar{B}$

Proof of Theorem 2.

(a) To prove that $\overline{A\cup B} = \bar A \cap \bar B$ we will show that each of the sets $\overline{A\cup B}$ and $\bar A \cap \bar B$ is a subset of the other.

First suppose $x \in \overline{A\cup B}$, then $x \notin A\cup B$. It means $x\notin A$ and $x \notin B$, or the same $x\in \bar A$ and $x \in \bar B$. It follows $x \in \bar A \cap \bar B$. Therefore, $\overline{A\cup B} \subset \bar A\cap\bar B$.

Now let us suppose that $x \in \bar A \cap \bar B$. Then $x\notin A$ and $x \notin B$, or the same $x \notin A\cup B$. It follows $x \in \overline{A\cup B}$. Therefore, $\bar A \cap \bar B \subset \overline{A\cup B}$.

Because we have $\overline{A\cup B} \subset \bar A\cap\bar B$ and $\bar A\cap\bar B \subset \overline{A\cup B}$ it follows that $\overline{A\cup B} = \bar A\cap\bar B$.

(b) By (a), $\overline{\bar A \cup \bar B} = \bar{\bar A} \cap \bar{\bar B}$, or $\overline{\bar A \cup \bar B} = A\cap B$, or $\bar A \cup \bar B = \overline{A\cap B}$, or the same, $\bar A \cup \bar B = \overline{A\cap B}$. □

Example: 10 Simplify the sets

(i) $A\cap(\overline{A\cap B})$

(ii) $(A\setminus B)\cap(A\cup B)$

Solution:

(i)
$$A\cap(\overline{A\cap B}) \underset{\substack{\uparrow \\ \text{Theorem 2}}}{=} A\cap(\bar A\cup\bar B) \underset{\substack{\uparrow \\ \text{Theorem 1}}}{=} (A\cap\bar A)\cup(A\ \ \bar B) \underset{\substack{\uparrow \\ \text{Theorem 1}}}{=} \emptyset\cup(A\setminus B) = A\setminus B$$

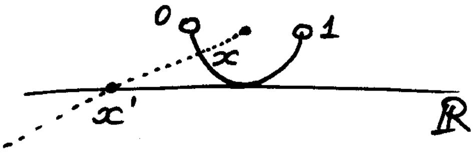
(ii) Since $A\setminus B \subset A\cup B$, then $(A\setminus B)\cap(A\cup B) = A\setminus B$.

Finite and Infinite Sets.

Definition. Let $f: A\to B$ be a function from A to B. The function f is called one-to-one if and only if different elements of A have different images and for any $b\in B$, $b=f(x)$ has a unique solution $x\in A$.

Examples: (11) Let $A = \{1, 2, 3, 4, \dots\}$ be the set of all natural numbers, $B = \{2, 4, 8, \dots\}$ be the set of all natural even numbers. Then $f(n) = 2n$ is one-to-one function from A to B.

(12) $A = (0, 1)$, $B = (-\infty, \infty) = \mathbb{R}$



The correspondence $x \rightarrow x'$ defines a one-to-one function $f(x) = x'$ between $(0, 1)$ and $(-\infty, \infty)$.

Definition. A finite set is a set which is either empty or in one-to-one correspondence with the set $\{1, 2, 3, \dots, n\}$ of the first n natural numbers for some $n \in \mathbb{N}$. Number n is said to be cardinality of the set.

(Note: We say "there is a one-to-one correspondence between A and B" if there is a one-to-one function f from A to B.)

If A is a finite set, the cardinality of A is the number of elements in A, this is denoted by $|A|$. Thus, $|\emptyset| = 0$, $|\{a, b, c, d\}| = 4$.

Definition. A set is said to be countably infinite if there is a one-to-one correspondence between the elements in the set and the elements in the set of all natural numbers $\mathbb{N}$.

$$\left.\begin{array}{l} \{0, 1, 2, 3, 4, \dots\} \\ \{2, 4, 6, 8, \dots\} \\ \{1, 3, 5, 7, \dots\} \\ \{3, 6, 9, 12, \dots\} \\ \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\} \end{array}\right\} \text{countably infinite sets}$$

Remark: If two finite sets A and B have the same cardinality - we write $|A| = |B|$ — then there is one-to-one correspondence between A and B. Conversely, if there is one-to-one function $f: A \to B$ then $|A| = |B|$.

This remark allows us to extend the notion of "same size".

Definition. Sets A and B have the same cardinality and we write $|A| = |B|$, if and only if there is a one-to-one function $f: A \longrightarrow B$.

Therefore, all countably infinite sets have the same cardinality. The symbol $\aleph_0$ (pronounced "aleph naught") has traditionally used to denote the cardinality of the natural numbers.

Example: 13 Show that $|\mathbb{Z}| = \aleph$, where $\mathbb{Z}$ is the set of all integers $= \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Solution.

The set of integers is infinite. To show that this set is countably infinite we list all integers:

$0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots\ldots$ This list is just

$f(1), f(2), f(3), f(4), f(5), \ldots$, where $f: \mathbb{N} \to \mathbb{Z}$ is defined as

$$f(n) = \begin{cases} \frac{1}{2} n, & \text{if } n \text{ is even}, \\ -\frac{1}{2}(n-1), & \text{if } n \text{ is odd}, \end{cases}$$

which is certainly one-to-one.

Problem. Show that $(0,1)$ is not countably infinite set.

Solution: The argument of this proof is an example of a proof by contradiction.

Suppose that $(0,1)$ is countably infinite set. Then there exists a list $a_1, a_2, a_3, \ldots\ldots$ of all the real numbers between 0 and 1. Write each of these numbers in decimal form. Our list would look like this:

$$a_1 = 0.\,a_{11}\,a_{12}\,a_{13}\,\ldots\ldots$$
$$a_2 = 0.\,a_{21}\,a_{22}\,a_{23}\,\ldots\ldots$$
$$a_3 = 0.\,a_{31}\,a_{32}\,a_{33}\,\ldots\ldots$$
$$\vdots$$

and remember, every real number in $(0,1)$ is supposed to be here. We can however to write down a number which is guaranteed not to be in the list.

Define for each $j = 1, 2, 3, \ldots$,

$$b_j = \begin{cases} 2, & \text{if } a_{jj} = 1 \\ 1, & \text{if } a_{jj} \neq 1 \end{cases}$$

Thus, b_j is always different from a_{jj}. Consider the number

$$b = 0.\,b_1\,b_2\,b_3\,\ldots\ldots$$

Then b is in the interval $(0,1)$, so it must be a_i for some i. But $b \neq a_1$ since b differs from a_1 in the first decimal place; $b \neq a_2$ since b differs from a_2 in the second decimal place and, generally $b \neq a_i$ since b differs from a_i in the ith decimal place. So, the hypothesis is false. The real numbers on $(0,1)$ are not countably infinite. $\square$

The "diagonal" argument we used in this proof is due to the great German mathematician Georg Cantor (1845-1918). It is ingenious and certainly not the kind of thing most of us would think up by ourselves. However, we have to remember that

- many infinite sets are countable ($\mathbb{N}, \mathbb{Z}, \mathbb{Q}$)
- but there are some sets that are much bigger ($\mathbb{R}$)

Note: Infinite sets come in different "sizes" just as do finite sets

HW. Problem: Let S be the set of all real numbers in the interval $(0,1)$ whose decimal expansions involve only 0 and 1. Prove that S is uncountable.