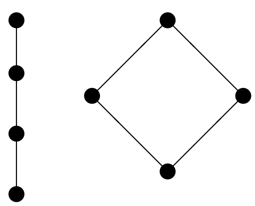
MATH Solution of MATH 110 Discrete Mathematics
MIDTERM II EXAM Date: April 23, 2005 , Time: 14:00-15:45

Problem 1. (a) Let $G = (V, E)$ be a loop-free undirected graph on n vertices. If G has 56 edges and $\overline{G}$ has 80 edges, what is n?
(b) Let $A = \{a, b, c, d\}$. How many different lattice do exist on A?
Solution. (a) Complete graph $K_n = (V, E_n)$ on n vertices has $\frac{n(n-1)}{2}$ edges. Note that the number of edges in K_n can also be evaluated as the sum of the number of edges in G and the number of edges in $\overline{G}$, i.e. $|E_n| = 56 + 80 = 136$ edges. We have $\frac{n(n-1)}{2} = 136$ that implies the number of vertices $n = 17$.
(b) We associate each lattice with its Hasse diagram. There are the following structures of Hasse diagram for lattices with 4 elements.



Therefore, there are $4! + 4 \times 3 = 36$ different latices on a set with 4 elements.
Problem 2. (a) Show that if any 15 integers are selected from the set

$$S = \{1, 2, 3, \dots, 26, 27\},$$

then among chosen numbers there are at least two whose sum is 28.
(b) Show that in any list of 35 natural numbers $a_1, a_2, \dots, a_{35}$, there is a string of consecutive items of the list, $a_l, a_{l+1}, a_{l+2}, \dots$, whose sum is divisible by 35.
Solution. (a) There are 13 pairs of numbers among numbers of S whose sum is exactly 28, namely $\{1, 27\}, \{2, 26\}, \dots, \{13, 15\}$. Denote these sets by H_1, $H_2, \dots$, H_{13} respectively. Denote by $H_{14} = \{14\}$. Now we choose 15 numbers (pigeons) among numbers of S, or the same we assign to each chosen number(pigeon) the set (the pigeonhole) it belongs to. Since the number of pigeons is strictly more than the number of pigeonholes then by the Pigeonhole Principle, there are two pigeons(two numbers) that will be assign to the same pigeonhole, or the same, sum up to 28.
(b) Denote by $x_i = a_1 + a_2 + \ldots + a_i$, $i = 1, 2, \ldots 35$. If one of these numbers is divisible by 35, say $35|x_j$, then the desired string is $a_1, a_2, \dots, a_j$. If none of them is divisible by 35 then after division by 35 all of them will have remainders that range from 1 to 34. Therefore, by the Pigeonhole principle (35 pigeons $x_1, \ldots, x_{35}$ and 34 pigeonholes (hole 1is for numbers $1+35k$, hole 2 is for numbers $2+35k$, …, hole 34 is for numbers $34+35k$, k is any integer)), two numbers will have the same remainder, say $x_i = r + 35k$ and $x_j = r + 35s$,$i > j$, k and s are integers, $1 \le r \le 34$. Then $x_i - x_j = 35(k - s) = a_{j+1} + \ldots + a_i$. It implies that the sum of elements in the string $a_{j+1}, a_{j+2}, \ldots a_i$ is divisible by 35.

Problem 3. A salesman, travelling by car, wishes to visit ten towns (including his own town) and return to home without passing through the same town twice. He knows that one, but only one, of the towns has a direct connection to each of the other towns (that is, connections passing through no other towns). He also knows that there are a total of 39 such direct road connections between pairs of towns. Why is he confident that he will be able to find such a route? Explain.

Solution. Consider graph $G = (V, E)$ with 10 vertices and 39 edges, where each vertex represents a town a salesman wishes to visit, and each edge between two vertices a and b means that there is a direct road connection between corresponding towns a and b. We have that there is a vertex, say $v_1 \in V$, such that $deg(v_1) = 9$ and for all other vertices, denote them by v_i, $i = 2, 3, \ldots, 10$, $deg(v_i) \leq 8$, $i = 2, 3, \ldots 10$. Let us show that G has a Hamiltonian cycle . It will follow then that a salesman can visit all towns and return home without passing through the same town twice.

We have $\sum_{v \in V} deg(v) = 78$. Then $\sum_{i=2}^{10} deg(v) = 78 - 9 = 69$. Assume that there is a vertex a such that $deg(a) < 5$ then $\sum_{i=2}^{10} deg(v) < 5 + 64 = 69$. It implies that $deg(v) \geq 5$ for all $v \in V$. Hence , $deg(x) + deg(y) \geq 10 = |V|$ for any pair of vertices x and y. Hence G has a Hamiltonian cycle.

Problem 4. A dog show is being judged from picture of the dogs. The judges would like to see pictures of the following pairs of dogs next to each other: Arfie and Fido, Arfie and Edgar, Arfie and Bowser, Bowser and Champ, Bowser and Edgar, Bowser and Dawg, Champ and Dawg, Dawg and Edgar, Dawg and Fido, Edgar and Fido, Fido and Goofy, Goofy and Dawg. Can the pictures be arranged in a row on the wall so that each desired pair of pictures appear together exactly once? If so, how? (There are many copies of each picture)

Solution. Consider a graph with 7 vertices A, B, C, D, E, F, G (each vertex we associate with a dog) and 12 edges (if we would like to see pictures of two dogs next to each other then we draw an edge between the corresponding vertices). The pictures can be arranged in a raw on the wall so that each desired pair appears together exactly once if and only if the constructed graph has an Euler trail. Since $deg(A) = 3$, $deg(D) = 5$ and all other vertices have even degrees, then our graph has an Euler trail. Moreover, any Euler trail starts either with A(then ends with D) or D(then ends with A). One possible example of an Euler trail (or desired arrangement of pictures) is

Arfie, Fido, Edgar, Arfie, Bowser, Edgar, Dawg, Bowser, Champ, Dawg, Fido, Googy, Dawg.