Math 110 Extra Exercises 4 (Solution).

Solution to Problem 1. We have, $(G, \circ)$ is a group, $H \cap K$ is a subset of G. It is enough to show that
1) $H \cap K$ is closed under the binary operation of G and
2) $\forall\, a \in H \cap K$, we have that $a^{-1} \in H \cap K$.

Proof of 1): Let $a, b \in H \cap K$. Then $a \in H$, $b \in H$, $a \in K$, $b \in$ Since H and K are subgroups of G, then $a \circ b \in H$ and $a \circ b \in K$. Therefore, $a \circ b \in H \cap K$.

Proof of 2): Let $a \in H \cap K$. Then $a \in H$ and $a \in K$. Since H and K are subgroups of G, then $a^{-1} \in H$ and $a^{-1} \in K$. Therefore, $a^{-1} \in H \cap K$.

Since $H \cap K$ possesses properties 1) and 2) then it is a a subgroup of G.

Solution to Problem 2. We have to prove that

$$(g \cdot f)(a \circ b) = \big((g \cdot f)(a)\big) \bullet \big((g \cdot f)(b)\big) \quad \forall\, a, b \in G.$$

We have

$$(g \cdot f)(a \circ b) = g(f(a \circ b)) \underset{f \text{ is a homomorphism}}{=} g(f(a) * f(b)) \underset{g \text{ is a homomorphism}}{=} g(f(a)) \bullet g(f(b)) =$$

$$= \big((g \cdot f)(a)\big) \bullet \big((g \cdot f)(b)\big) \quad \forall\, a, b \in G.$$

Then $g \cdot f$ is a homomorphism from G to K.

Solution to Problem 3. We have $a^5 = e$. Then

$$a \circ b = a^5 \circ (a \circ b) = a^3 \circ (a^3 \circ b) = a^3 \circ (b \circ a^3) = (a^3 \circ b) \circ a^3 = (b \circ a^3) \circ a^3 =$$

$$= (b \circ a) \circ a^5 = b \circ a, \text{ i.e. } a \circ b = b \circ a.$$

Solution to Problem 4. Let $\operatorname{ord}(a \circ b) = n$. Then

$$\underbrace{(a \circ b) \circ (a \circ b) \circ \dots \circ (a \circ b)}_{n \text{ times}} = e \Rightarrow a \circ (b \circ a) \circ (b \circ a) \circ \dots + (b \circ a) \circ b = e$$

$$\underset{\text{multiply by } b \text{ from the left}}{\Rightarrow} (b \circ a) \circ (b \circ a) \circ \dots \circ (b \circ a) \circ b = b \circ e = b \underset{\text{cancel } b \text{ from the right}}{\Rightarrow} (b \circ a)^n = e$$

$\Rightarrow \operatorname{ord}(b \circ a) \le n = \operatorname{ord}(a \circ b)$. We have proved that $\operatorname{ord}(b \circ a) \le \operatorname{ord}(a \circ b)$ $\forall\, a, b \in G$. Similarly, we can show that $\operatorname{ord}(a \circ b) \le \operatorname{ord}(b \circ a)$ $\forall\, a, b \in G$. Hence, $\operatorname{ord}(a \circ b) = \operatorname{ord}(b \circ a)$.