SAMPLE

MATH 116-02 QUIZ 12

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Problem 1. Let $\mathbf{F} = (2xy + z^2)\vec{i} + (x^2 + 2yz + e^z)\vec{j} + (y^2 + 2xz + ye^z)\vec{k}$ be a vector field in the xyz-space.
(i) Prove that $\mathbf{F}$ is conservative.
(ii) Find its potential function.
(iii) Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is any smooth path joining $A = (1, 0, 2)$ to $B = (3, 4, 0)$.

Solution: $F = M\vec{i} + N\vec{j} + P\vec{k}$, where $M = 2xy + z^2$, $N = x^2 + 2yz + e^z$, $P = y^2 + 2xz + ye^z$.

(i) It is enough to check that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}.$$

We have,

$$\left.\begin{matrix} \frac{\partial M}{\partial y} = 2x \\ \frac{\partial N}{\partial x} = 2x \end{matrix}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad \left.\begin{matrix} \frac{\partial M}{\partial z} = 2z \\ \frac{\partial P}{\partial x} = 2z \end{matrix}\right\} \Rightarrow \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \left.\begin{matrix} \frac{\partial N}{\partial z} = 2y + e^z \\ \frac{\partial P}{\partial y} = 2y + e^z \end{matrix}\right\} \Rightarrow \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}.$$

Therefore, F is a conservative vector field.

(ii) It follows from part (i) that $F = \nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$ for some function f. Then

$$\frac{\partial f}{\partial x} = M = 2xy + z^2 \Rightarrow f(x, y, z) = x^2y + xz^2 + g(y, z)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2y + xz^2 + g(y, z)) = x^2 + \frac{\partial g(y,z)}{\partial y} = N = x^2 + 2yz + e^z \Rightarrow \frac{\partial g(y,z)}{\partial y} = 2yz + e^z$$

$$\Rightarrow g(y, z) = y^2z + ye^z + h(z) \Rightarrow f(x, y, z) = x^2y + xz^2 + y^2z + ye^z + h(z)$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(x^2y + xz^2 + y^2z + ye^z + h(z)) = 2xz + y^2 + ye^z + \frac{dh(z)}{dz} = P = y^2 + 2xz + ye^z$$

$$\Rightarrow \frac{dh(z)}{dz} = 0 \Rightarrow h(z) = Const \Rightarrow f(x, y, z) = x^2y + xz^2 + y^2z + ye^z + Const$$

Hence, any potential function of a vector field F is $f(x, y, z) = yx^2 + xz^2 + y^2z + ye^z + Const$.

(iii) $$\int_C F \cdot dr = \int_A^B \nabla f \cdot dr = f(B) - f(A) = f(3, 4, 0) - f(1, 0, 2) =$$

$$= 40 + C - (4 + C) = 36.$$