Solution to MATH 116: INTERMEDIATE CALCULUS III
FINAL Exam, July 19, 2006

1. Find the points on the sphere $(x-1)^2+(y+2)^2+z^2=2$ closest and farthest to the point $Q=(3,1,-1)$.

Solution: We have to find the points whose coordinates minimize and maximize the values of the function

$$f(x,y,z)=(x-3)^2+(y-1)^2+(z+1)^2$$

subject to the constraint equation

$$g(x,y,z)=(x-1)^2+(y+2)^2+z^2-2=0.$$

To do so, we find the values of x, y, z and λ for which

$$\nabla f=\lambda\nabla g \qquad \textit{and} \qquad g(x,y,z)=0$$

The gradient equation gives us

$$\begin{cases} 2(x-3)=2\lambda(x-1) & & x(1-\lambda)=3-\lambda & & x=\dfrac{3-\lambda}{1-\lambda} \\ 2(y-1)=2\lambda(y+2) & \Longrightarrow & y(1-\lambda)=1+2\lambda & \Longrightarrow & y=\dfrac{1+2\lambda}{1-\lambda} \\ 2(z+1)=2\lambda z & & z(1-\lambda)=-1 & & z=-\dfrac{1}{1-\lambda} \end{cases}$$

We substitute these expressions for x, y, z into $g(x,y,z)=0$ to obtain:

$$2=\left(\frac{3-\lambda}{1-\lambda}-1\right)^2+\left(\frac{1+2\lambda}{1-\lambda}+2\right)^2+\frac{1}{(1-\lambda)^2}=\frac{14}{(1-\lambda)^2}\Longrightarrow(1-\lambda)^2=7\Longrightarrow\lambda=1\pm\sqrt{7}.$$

For $\lambda=1+\sqrt{7}$ the corresponding point on the sphere is point A$\left(1-\frac{2}{\sqrt{7}};-2-\frac{3}{\sqrt{7}};\frac{1}{\sqrt{7}}\right)$, for $\lambda=1-\sqrt{7}$ the corresponding point on the sphere is point B$\left(1+\frac{2}{\sqrt{7}};-2+\frac{3}{\sqrt{7}};-\frac{1}{\sqrt{7}}\right)$. To decide which point is the closest and which one is the farthest to point Q, we write the values of function $f(x,y,z)$ at points A and B:

$$f|_A=f\left(1-\frac{2}{\sqrt{7}};-2-\frac{3}{\sqrt{7}};\frac{1}{\sqrt{7}}\right)=\left(-2-\frac{2}{\sqrt{7}}\right)^2+\left(-3-\frac{3}{\sqrt{7}}\right)^2+\left(1+\frac{1}{\sqrt{7}}\right)^2,$$

$$f|_B=f\left(1+\frac{2}{\sqrt{7}};-2+\frac{3}{\sqrt{7}};-\frac{1}{\sqrt{7}}\right)=\left(-2+\frac{2}{\sqrt{7}}\right)^2+\left(-3+\frac{3}{\sqrt{7}}\right)^2+\left(1-\frac{1}{\sqrt{7}}\right)^2,$$

Clearly, $f|_A > f|_B$. Therefore, on the sphere the farthest point to point Q is point A$\left(1-\frac{2}{\sqrt{7}};-2-\frac{3}{\sqrt{7}};\frac{1}{\sqrt{7}}\right)$ and the closest point to point Q is point B$\left(1+\frac{2}{\sqrt{7}};-2+\frac{3}{\sqrt{7}};-\frac{1}{\sqrt{7}}\right)$.

2. a) Let f be a function defined by

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$$

Prove that the above function f is continuous everywhere in $\mathbf{R^2}$.

Solution: Function $f(x,y)$ is continuous at any point $(x,y) \neq (0,0)$ since $f(x,y)$ is a rational function of x, y with non vanishing denominator x^2+y^2 at any $(x,y) \neq (0,0)$. To show that $f(x,y)$ is continuous at $(0,0)$ it is necessary to check that

1) f is defined at $(0,0)$;
2) $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists;
3) $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$.

Function f is defined at $(0,0)$ by value 0, i.e. $f(0,0) = 0$. Hence property 1) is satisfied. Clearly,

$$\left|\frac{x^2y}{x^2+y^2}\right| \leq \frac{x^2|y|}{x^2} = |y|,$$

and hence,

$$-|y| \leq \frac{x^2y}{x^2+y^2} \leq |y|.$$

Since also $\lim\limits_{(x,y)\to(0,0)} |y| = \lim\limits_{(x,y)\to(0,0)} -|y| = 0$ then by Sandwich Theorem,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0 = f(0,0).$$

It means that properties 2) and 3) also hold. Thus, function $f(x,y)$ is continuous at $(0,0)$.

2. b) Are the first partial derivatives of the function $f(x,y)$ in part (a) continuous at the origin $(0,0)$? Explain your answer.

Solution: For any $(x,y) \neq (0,0)$, we have

$$\frac{\partial f}{\partial x} = \frac{2xy(x^2+y^2) - 2x(x^2y)}{(x^2+y^2)^2} = \frac{2xy^3}{(x^2+y^2)^2},$$

$$\frac{\partial f}{\partial y} = \frac{x^2(x^2+y^2) - 2y(x^2y)}{(x^2+y^2)^2} = \frac{x^4 - x^2y^2}{(x^2+y^2)^2}.$$

Since

$$\lim_{y=kx, x\to 0} \frac{\partial f}{\partial x} = \lim_{y=kx, x\to 0} \frac{2xy^3}{(x^2+y^2)^2} = \frac{2k^3}{(1+k^2)^2}$$

depends on k, then $\lim\limits_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}$ does not exist. Therefore, $\frac{\partial f}{\partial x}$ is not continuous at $(0,0)$ (note that $\frac{\partial f}{\partial x}|_{(0,0)} = 0$).

Since

$$\lim_{y=kx, x\to 0} \frac{\partial f}{\partial y} = \lim_{y=kx, x\to 0} \frac{x^4 - x^2y^2}{(x^2+y^2)^2} = \frac{1-k^2}{(1+k^2)^2}$$

depends on k, then $\lim\limits_{(x,y)\to(0,0)} \frac{\partial f}{\partial y}$ does not exist. Therefore, $\frac{\partial f}{\partial y}$ is not continuous at $(0,0)$ (note that $\frac{\partial f}{\partial y}|_{(0,0)} = 0$).

3. a) Find the volume of the region D which is bounded
below by the plane $z = 0$,
above by the plane $z = 4$,
on the sides by the surface $y = x^3$ and two planes $x = 1$ and $y = 0$.
Solution:

$$V = \iiint_D dV = \int_0^1 \int_0^{x^3} \int_0^4 dzdydx = \int_0^1 \int_0^{x^3} 4dydx = \int_0^1 4x^3dx = 1.$$

3. b) Evaluate the integral $\iiint_R z^{-4}\, dV$ over the *unbounded* region R in the first octant ($x > 0, y > 0, z > 0$) that lies between the cones $z^2 = 3(x^2 + y^2)$, $3z^2 = x^2 + y^2$ and outside the sphere $x^2 + y^2 + z^2 = 1$.
Solution: In spherical coordinates the boundary surfaces $z^2 = 3(x^2+y^2)$, $3z^2 = x^2+y^2$, $x^2+y^2+z^2 = 1$ have the equations $\varphi = \frac{\pi}{6}$, $\varphi = \frac{\pi}{3}$ and $\rho = 1$ respectively. Therefore,

$$\iiint_R z^{-4}\, dV = \int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_1^{\infty} \frac{\rho^2 \sin\varphi}{\rho^4 \cos^4\varphi} d\rho d\varphi d\theta = \int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin\varphi}{\cos^4\varphi} d\varphi d\theta = \int_0^{\frac{\pi}{2}} \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{t^4} dt d\theta = \frac{4\pi}{9\sqrt{3}}(3\sqrt{3} - 1).$$

4. a) Find the counterclockwise circulation of the field

$$\mathbf{F} = (3x^2y + x^3 \sin^2 x)\mathbf{i} + (x^3 + y^3 + 2x + 3y)\mathbf{j}$$

around the triangle with vertices $(0,0)$, $(2,0)$ and $(1,1)$.
Solution: Denote by R a plane triangular region with vertices $(0,0)$, $(2,0)$ and $(1,1)$. By Green's Theorem, the counterclockwise circulation of the field $\mathbf{F}$ around the boundary of R is

$$\oint_C F \cdot dr = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_R (3x^2 + 2 - 3x^2) dA = 2\iint_R dA = 2Area(R) = 2 \times 1 = 2.$$

4. b) Let $\mathbf{F} = (2xy + z^2)\mathbf{i} + (x^2 + 2yz + e^z)\mathbf{j} + (y^2 + 2xz + y\,e^z)\mathbf{k}$be a vector field in $\mathbb{R}^3$. **(i)** Prove that $\mathbf{F}$ is conservative. **(ii)** Find its potential function. **(iii)** Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is a any smooth path joining $A = (1,0,2)$ to $B = (3,4,0)$.
Solution: (i) We have,

$$curlF = \nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2xy + z^2) & (x^2 + 2yz + e^z) & (y^2 + 2xz + y\,e^z) \end{vmatrix} =$$

$$= \vec{i}(\frac{\partial}{\partial y}(y^2 + 2xz + y\,e^z) - \frac{\partial}{\partial z}(x^2 + 2yz + e^z)) - \vec{j}(\frac{\partial}{\partial x}(y^2 + 2xz + y\,e^z) - \frac{\partial}{\partial z}(2xy + z^2)) +$$

$$+\vec{k}(\frac{\partial}{\partial x}(x^2 + 2yz + e^z) - \frac{\partial}{\partial y}(2xy + z^2)) = 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{\mathbf{0}}.$$

Since all components of vector field $\mathbf{F}$ have continuous first order partial derivatives in $\mathbb{R}^3$ and $curlF = \nabla \times F = \vec{\mathbf{0}}$ in $\mathbb{R}^3$ then $\mathbf{F}$ is a conservative vector field in $\mathbb{R}^3$.
(ii) Since $\mathbf{F}$ is conservative then for some function f(called a potential function of $\mathbf{F}$):

$$\mathbf{F} = (2xy + z^2)\vec{i} + (x^2 + 2yz + e^z)\vec{j} + (y^2 + 2xz + y\,e^z)\vec{k} = \nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

Hence,

$$2xy + z^2 = \frac{\partial f}{\partial x} \qquad \rightarrow f(x,y,z) = x^2y + z^2x + g(y,z)$$

$$x^2 + 2yz + e^z = \frac{\partial f}{\partial y} = x^2 + \frac{\partial g(y,z)}{\partial y} \qquad \rightarrow g(y,z) = y^2z + ye^z + h(z),$$
$$f(x,y,z) = x^2y + z^2x + y^2z + ye^z + h(z)$$

$$y^2 + 2xz + ye^z = \frac{\partial f}{\partial z} = 2zx + y^2 + ye^z + \frac{\partial h(z)}{\partial z} \qquad \rightarrow h(z) = Const$$
$$f(x,y,z) = x^2y + z^2x + y^2z + ye^z + Const$$

Hence, any potential function of vector field $\mathbf{F}$ is $f(x,y,z) = x^2y + z^2x + y^2z + ye^z + Const$.

(iii) $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_A^B \nabla f \cdot dr = f(B) - f(A) = f(3,4,0) - f(1,0,2) = 40 + C - (4 + C) = 36$$

5. a) Find the area of the cap cut from the hemisphere $x^2 + y^2 + z^2 = 2, z \geq 0$, by the cylinder $x^2 + y^2 = 1$.
Solution: see the solution of Example 2 on page 1184 from the Textbook.

5. b) Use the Divergence Theorem to evaluate the outward flux of

$$\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j} + x^3y^3\mathbf{k}$$

through the surface of the tetrahedron D bounded by the plane $x + y + z = 1$ and the coordinate planes $x = 0,\ y = 0,\ z = 0$.
Solution: We have that $divF = 2x + x = 3x$. By the Divergence Theorem,

$$Outward \quad Flux = \iiint_D divF\, dV = \iiint_D 3x dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 3xdzdydx = \frac{1}{8}.$$