

**Solution to MATH 116: INTERMEDIATE CALCULUS III**  
**FINAL Exam, July 19, 2006**

1. Find the points on the sphere  $(x - 1)^2 + (y + 2)^2 + z^2 = 2$  closest and farthest to the point  $Q = (3, 1, -1)$ .

**Solution:** We have to find the points whose coordinates minimize and maximize the values of the function

$$f(x, y, z) = (x - 3)^2 + (y - 1)^2 + (z + 1)^2$$

subject to the constraint equation

$$g(x, y, z) = (x - 1)^2 + (y + 2)^2 + z^2 - 2 = 0.$$

To do so, we find the values of  $x, y, z$  and  $\lambda$  for which

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x, y, z) = 0$$

The gradient equation gives us

$$\begin{cases} 2(x - 3) = 2\lambda(x - 1) & x(1 - \lambda) = 3 - \lambda & x = \frac{3 - \lambda}{1 - \lambda} \\ 2(y - 1) = 2\lambda(y + 2) \implies y(1 - \lambda) = 1 + 2\lambda \implies & y = \frac{1 + 2\lambda}{1 - \lambda} \\ 2(z + 1) = 2\lambda z & z(1 - \lambda) = -1 & z = -\frac{1}{1 - \lambda} \end{cases}$$

We substitute these expressions for  $x, y, z$  into  $g(x, y, z) = 0$  to obtain:

$$2 = \left(\frac{3 - \lambda}{1 - \lambda} - 1\right)^2 + \left(\frac{1 + 2\lambda}{1 - \lambda} + 2\right)^2 + \frac{1}{(1 - \lambda)^2} = \frac{14}{(1 - \lambda)^2} \implies (1 - \lambda)^2 = 7 \implies \lambda = 1 \pm \sqrt{7}.$$

For  $\lambda = 1 + \sqrt{7}$  the corresponding point on the sphere is point A  $\left(1 - \frac{2}{\sqrt{7}}; -2 - \frac{3}{\sqrt{7}}; \frac{1}{\sqrt{7}}\right)$ , for  $\lambda = 1 - \sqrt{7}$  the corresponding point on the sphere is point B  $\left(1 + \frac{2}{\sqrt{7}}; -2 + \frac{3}{\sqrt{7}}; -\frac{1}{\sqrt{7}}\right)$ . To decide which point is the closest and which one is the farthest to point Q, we write the values of function  $f(x, y, z)$  at points A and B:

$$f|_A = f\left(1 - \frac{2}{\sqrt{7}}; -2 - \frac{3}{\sqrt{7}}; \frac{1}{\sqrt{7}}\right) = \left(-2 - \frac{2}{\sqrt{7}}\right)^2 + \left(-3 - \frac{3}{\sqrt{7}}\right)^2 + \left(1 + \frac{1}{\sqrt{7}}\right)^2,$$

$$f|_B = f\left(1 + \frac{2}{\sqrt{7}}; -2 + \frac{3}{\sqrt{7}}; -\frac{1}{\sqrt{7}}\right) = \left(-2 + \frac{2}{\sqrt{7}}\right)^2 + \left(-3 + \frac{3}{\sqrt{7}}\right)^2 + \left(1 - \frac{1}{\sqrt{7}}\right)^2,$$

Clearly,  $f|_A > f|_B$ . Therefore, on the sphere the farthest point to point Q is point A  $\left(1 - \frac{2}{\sqrt{7}}; -2 - \frac{3}{\sqrt{7}}; \frac{1}{\sqrt{7}}\right)$  and the closest point to point Q is point B  $\left(1 + \frac{2}{\sqrt{7}}; -2 + \frac{3}{\sqrt{7}}; -\frac{1}{\sqrt{7}}\right)$ .

2. a) Let  $f$  be a function defined by

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{otherwise} \end{cases}$$

Prove that the above function  $f$  is continuous everywhere in  $\mathbf{R}^2$ .

**Solution:** Function  $f(x, y)$  is continuous at any point  $(x, y) \neq (0, 0)$  since  $f(x, y)$  is a rational function of  $x, y$  with non vanishing denominator  $x^2 + y^2$  at any  $(x, y) \neq (0, 0)$ . To show that  $f(x, y)$  is continuous at  $(0, 0)$  it is necessary to check that

- 1)  $f$  is defined at  $(0, 0)$ ;
- 2)  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  exists;
- 3)  $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = f(0, 0)$ .

Function  $f$  is defined at  $(0, 0)$  by value 0, i.e.  $f(0, 0) = 0$ . Hence property 1) is satisfied.

Clearly,

$$\left| \frac{x^2 y}{x^2 + y^2} \right| \leq \frac{x^2 |y|}{x^2} = |y|,$$

and hence,

$$-|y| \leq \frac{x^2 y}{x^2 + y^2} \leq |y|.$$

Since also  $\lim_{(x,y) \rightarrow (0,0)} |y| = \lim_{(x,y) \rightarrow (0,0)} -|y| = 0$  then by Sandwich Theorem,

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^2 + y^2} = 0 = f(0, 0).$$

It means that properties 2) and 3) also hold. Thus, function  $f(x, y)$  is continuous at  $(0, 0)$ .

2. b) Are the first partial derivatives of the function  $f(x, y)$  in part (a) continuous at the origin  $(0, 0)$ ? Explain your answer.

**Solution:** For any  $(x, y) \neq (0, 0)$ , we have

$$\frac{\partial f}{\partial x} = \frac{2xy(x^2 + y^2) - 2x(x^2 y)}{(x^2 + y^2)^2} = \frac{2xy^3}{(x^2 + y^2)^2},$$

$$\frac{\partial f}{\partial y} = \frac{x^2(x^2 + y^2) - 2y(x^2 y)}{(x^2 + y^2)^2} = \frac{x^4 - x^2 y^2}{(x^2 + y^2)^2}.$$

Since

$$\lim_{y=kx, x \rightarrow 0} \frac{\partial f}{\partial x} = \lim_{y=kx, x \rightarrow 0} \frac{2xy^3}{(x^2 + y^2)^2} = \frac{2k^3}{(1 + k^2)^2}$$

depends on  $k$ , then  $\lim_{(x,y) \rightarrow (0,0)} \frac{\partial f}{\partial x}$  does not exist. Therefore,  $\frac{\partial f}{\partial x}$  is not continuous at  $(0, 0)$  (note that  $\frac{\partial f}{\partial x}|_{(0,0)} = 0$ ).

Since

$$\lim_{y=kx, x \rightarrow 0} \frac{\partial f}{\partial y} = \lim_{y=kx, x \rightarrow 0} \frac{x^4 - x^2 y^2}{(x^2 + y^2)^2} = \frac{1 - k^2}{(1 + k^2)^2}$$

depends on  $k$ , then  $\lim_{(x,y) \rightarrow (0,0)} \frac{\partial f}{\partial y}$  does not exist. Therefore,  $\frac{\partial f}{\partial y}$  is not continuous at  $(0, 0)$  (note that  $\frac{\partial f}{\partial y}|_{(0,0)} = 0$ ).

**3. a)** Find the volume of the region  $D$  which is bounded below by the plane  $z = 0$ , above by the plane  $z = 4$ , on the sides by the surface  $y = x^3$  and two planes  $x = 1$  and  $y = 0$ .

**Solution:**

$$V = \int \int \int_D dV = \int_0^1 \int_0^{x^3} \int_0^4 dz dy dx = \int_0^1 \int_0^{x^3} 4 dy dx = \int_0^1 4x^3 dx = 1.$$

**3. b)** Evaluate the integral  $\int \int \int_R z^{-4} dV$  over the *unbounded* region  $R$  in the first octant ( $x > 0, y > 0, z > 0$ ) that lies between the cones  $z^2 = 3(x^2 + y^2)$ ,  $3z^2 = x^2 + y^2$  and outside the sphere  $x^2 + y^2 + z^2 = 1$ .

**Solution:** In spherical coordinates the boundary surfaces  $z^2 = 3(x^2 + y^2)$ ,  $3z^2 = x^2 + y^2$ ,  $x^2 + y^2 + z^2 = 1$  have the equations  $\varphi = \frac{\pi}{6}$ ,  $\varphi = \frac{\pi}{3}$  and  $\rho = 1$  respectively. Therefore,

$$\int \int \int_R z^{-4} dV = \int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_1^{\infty} \frac{\rho^2 \sin \varphi}{\rho^4 \cos^4 \varphi} d\rho d\varphi d\theta = \int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin \varphi}{\cos^4 \varphi} d\varphi d\theta = \int_0^{\frac{\pi}{2}} \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{t^4} dt d\theta = \frac{4\pi}{9\sqrt{3}}(3\sqrt{3} - 1).$$

4. a) Find the counterclockwise circulation of the field

$$\mathbf{F} = (3x^2y + x^3 \sin^2 x)\mathbf{i} + (x^3 + y^3 + 2x + 3y)\mathbf{j}$$

around the triangle with vertices  $(0, 0)$ ,  $(2, 0)$  and  $(1, 1)$ .

**Solution:** Denote by  $R$  a plane triangular region with vertices  $(0, 0)$ ,  $(2, 0)$  and  $(1, 1)$ . By Green's Theorem, the counterclockwise circulation of the field  $\mathbf{F}$  around the boundary of  $R$  is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int \int_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \int \int_R (3x^2 + 2 - 3x^2) dA = 2 \int \int_R dA = 2 \text{Area}(R) = 2 \times 1 = 2.$$

4. b) Let  $\mathbf{F} = (2xy + z^2)\mathbf{i} + (x^2 + 2yz + e^z)\mathbf{j} + (y^2 + 2xz + ye^z)\mathbf{k}$  be a vector field in  $\mathbb{R}^3$ . (i) Prove that  $\mathbf{F}$  is conservative. (ii) Find its potential function. (iii) Evaluate the integral  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  $C$  is a any smooth path joining  $A = (1, 0, 2)$  to  $B = (3, 4, 0)$ .

**Solution:** (i) We have,

$$\begin{aligned} \text{curl} \mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2xy + z^2) & (x^2 + 2yz + e^z) & (y^2 + 2xz + ye^z) \end{vmatrix} = \\ &= \vec{i} \left( \frac{\partial}{\partial y} (y^2 + 2xz + ye^z) - \frac{\partial}{\partial z} (x^2 + 2yz + e^z) \right) - \vec{j} \left( \frac{\partial}{\partial x} (y^2 + 2xz + ye^z) - \frac{\partial}{\partial z} (2xy + z^2) \right) + \\ &\quad + \vec{k} \left( \frac{\partial}{\partial x} (x^2 + 2yz + e^z) - \frac{\partial}{\partial y} (2xy + z^2) \right) = 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{0}. \end{aligned}$$

Since all components of vector field  $\mathbf{F}$  have continuous first order partial derivatives in  $\mathbb{R}^3$  and  $\text{curl} \mathbf{F} = \nabla \times \mathbf{F} = \vec{0}$  in  $\mathbb{R}^3$  then  $\mathbf{F}$  is a conservative vector field in  $\mathbb{R}^3$ .

(ii) Since  $\mathbf{F}$  is conservative then for some function  $f$  (called a potential function of  $\mathbf{F}$ ):

$$\mathbf{F} = (2xy + z^2)\vec{i} + (x^2 + 2yz + e^z)\vec{j} + (y^2 + 2xz + ye^z)\vec{k} = \nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

Hence,

$$2xy + z^2 = \frac{\partial f}{\partial x} \quad \rightarrow f(x, y, z) = x^2y + z^2x + g(y, z)$$

$$\begin{aligned} x^2 + 2yz + e^z &= \frac{\partial f}{\partial y} = x^2 + \frac{\partial g(y, z)}{\partial y} & \rightarrow g(y, z) &= y^2z + ye^z + h(z), \\ & & f(x, y, z) &= x^2y + z^2x + y^2z + ye^z + h(z) \end{aligned}$$

$$\begin{aligned} y^2 + 2xz + ye^z &= \frac{\partial f}{\partial z} = 2zx + y^2 + ye^z + \frac{\partial h(z)}{\partial z} & \rightarrow h(z) &= \text{Const} \\ & & f(x, y, z) &= x^2y + z^2x + y^2z + ye^z + \text{Const} \end{aligned}$$

Hence, any potential function of vector field  $\mathbf{F}$  is  $f(x, y, z) = x^2y + z^2x + y^2z + ye^z + \text{Const}$ .

$$(iii) \int_C \mathbf{F} \cdot d\mathbf{r} = \int_A^B \nabla f \cdot d\mathbf{r} = f(B) - f(A) = f(3, 4, 0) - f(1, 0, 2) = 40 + C - (4 + C) = 36$$

5. a) Find the area of the cap cut from the hemisphere  $x^2 + y^2 + z^2 = 2, z \geq 0$ , by the cylinder  $x^2 + y^2 = 1$ .

**Solution:** see the solution of Example 2 on page 1184 from the Textbook.

5. b) Use the Divergence Theorem to evaluate the outward flux of

$$\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j} + x^3y^3\mathbf{k}$$

through the surface of the tetrahedron  $D$  bounded by the plane  $x + y + z = 1$  and the coordinate planes  $x = 0, y = 0, z = 0$ .

**Solution:** We have that  $\text{div}F = 2x + x = 3x$ . By the Divergence Theorem,

$$\text{Outward Flux} = \int \int \int_D \text{div}F \, dV = \int \int \int_D 3x \, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 3x \, dz \, dy \, dx = \frac{1}{8}.$$