Solution to MATH 116 MIDTERM II EXAM

Problem 1. Let R be a plane region in the first quadrant bounded by the lines $y = 7 - x$, $7y = 7 - x$, and the parabola $y = 1 + x^2$. Set up the double integral

$$I = \iint\limits_R f(x, y)dA$$

in Cartesian coordinates using the following orders of integration.

(a) $dxdy$

(b) $dydx$

(c) Integrate function $f(x, y) = \frac{1}{\sqrt{y}}$ over the region R.

Solution. (a) Let $A(0, 1)$, $B(2, 5)$ and $C(7, 0)$. The boundaries of the region R are two line segments AC, BC and part of the parabola $y = x^2 + 1$ from point A to point B.

(a) To write limits of integration of integral I with $dxdy$ order, we divide our region into two parts by line $y = 1$. Then

$$I = \int_1^5 \int_{\sqrt{y-1}}^{7-y} f(x, y)dxdy + \int_0^1 \int_{7-7y}^{7-y} f(x, y)dxdy$$

(b)To write limits of integration of integral I with $dydx$ order, we divide our region into two parts by line $x = 2$. Then

$$I = \int_0^2 \int_{\frac{7-x}{7}}^{x^2+1} f(x, y)dydx + \int_2^7 \int_{\frac{7-x}{7}}^{7-x} f(x, y)dydx$$

$$(c) \quad \iint\limits_R \frac{1}{\sqrt{y}}dA = \int_0^2 \int_{\frac{7-x}{7}}^{x^2+1} \frac{1}{\sqrt{y}}dydx + \int_2^7 \int_{\frac{7-x}{7}}^{7-x} \frac{1}{\sqrt{y}}dydx = \int_0^2 2\sqrt{y}\Big|_{y=1-\frac{x}{7}}^{y=x^2+1} dx + \int_2^7 2\sqrt{y}\Big|_{y=1-\frac{x}{7}}^{y=7-x} dx =$$

$$= 2\int_0^2 (\sqrt{x^2+1} - \sqrt{1 - \frac{x}{7}})dx + 2\int_2^7 (\sqrt{7-x} - \sqrt{1 - \frac{x}{7}})dx =$$

$$= 2((\frac{x}{2}\sqrt{x^2+1} + \frac{1}{2}\ln(x + \sqrt{x^2+1}))|_0^2 + \frac{14}{3}(1 - \frac{x}{7})^{3/2}|_0^7 - \frac{2}{3}(7-x)^{3/2}|_2^7) = \frac{26\sqrt{5} - 28}{3} + \ln(2 + \sqrt{5})$$

Remark: To evaluate $\int_0^2 \sqrt{x^2+1}dx$, use trigonometric substitution $x = \tan t$

Problem 2. Let R be the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$. Evaluate the integral

$$I = \iint\limits_R e^{(y-x)/(x+y)}dA$$

Solution. Let

$$\begin{matrix} u = y - x \\ v = x + y \end{matrix} \quad \rightarrow \quad \begin{matrix} x = \frac{1}{2}u + \frac{1}{2}v \\ y = -\frac{1}{2}u + \frac{1}{2}v \end{matrix} \quad \rightarrow \quad J(u, v) = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

The boundaries in the xy-plane are the lines $x = 0, y = 0, y + x = 1$. They are transformed into the boundaries in the uv-plane: $u = v, u = -v, v = 1$. Therefore, the uv-region of integration is the triangle with vertices $(0,0)$, $(1,1)$ and $(-1,1)$. Then

$$I=\int_0^1\int_{-v}^{v} e^{u/v}|J(u,v)|dudv=\frac{1}{2}\int_0^1\int_{-v}^{v} e^{u/v}dudv=\frac{1}{2}\int_0^1 ve^{u/v}|_{u=-v}^{u=v}dv=\frac{1}{2}\int_0^1 v(e-e^{-1})dv=\frac{e-e^{-1}}{4}$$

Problem 3. (a) Sketch the solid D between the graphs of $z=0$, $z=\sqrt{1-y}$ and lying above the plane region R which is bounded by $y=\sqrt{x-1}$, $y=0$, $y=1$, and $x=0$.

(b) Evaluate the volume V of the solid D described in part (a).

Solution only to part (b).

$$V=\iint_R \sqrt{1-y}dA=\int_0^1\int_0^{1+y^2}\sqrt{1-y}dxdy=\int_0^1(1+y^2)\sqrt{1-y}dy=\int_0^1 2t^{1/2}-2t^{3/2}+t^{5/2}dt=\frac{86}{105}$$

Problem 4. Consider integral

$$I=\int_0^{2\pi}\int_0^2\int_0^{\sqrt{9-r^2}} r^2dzdrd\theta$$

(a) Write integral I in rectangular coordinates. (Do not evaluate the integral).

(b) Write integral I in spherical coordinates. (Do not evaluate the integral).

Solution. (a) The region of integration is a solid bounded below by the plane $z=0$, above by the sphere $x^2+y^2+z^2=9$ and on sides by the cylinder $x^2+y^2=4$. Note that $r=\sqrt{x^2+y^2}$. Then

$$I=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{\sqrt{9-x^2-y^2}}\sqrt{x^2+y^2}dzdydx$$

(b) To write integral I in spherical coordinates we divide the region of integration into two parts. The first part is a cylinder $x^2+y^2=4$ (its equation in spherical coordinates is $\rho\sin\phi=2$) between the plane $z=0$ and the cone $\phi=tan^{-1}\frac{2}{\sqrt{5}}$. The second part is sphere $\rho=3$ above the cone $\phi=tan^{-1}\frac{2}{\sqrt{5}}$. Then

$$I=\int_0^{2\pi}\int_{\tan^{-1}\frac{2}{\sqrt{5}}}^{\pi/2}\int_0^{\frac{2}{\sin\phi}}\rho^3\sin^2\phi d\rho d\phi d\theta+\int_0^{2\pi}\int_0^{\tan^{-1}\frac{2}{\sqrt{5}}}\int_0^3\rho^3\sin^2\phi d\rho d\phi d\theta$$

Problem 5. Find the work done by the force $F=y\vec{i}+(y^2+z)\vec{j}+(x^2-z)\vec{k}$ over the curve C of intersection of the surfaces $z=x^2$ and $x^2+y^2=4$.

Solution. Parametrization of the curve C is $x=2\cos t, y=2\sin t, z=4\cos^2 t, 0\le t\le 2\pi$. Then the curve is the graph of function $r(t)=2\cos t\vec{i}+2\sin t\vec{j}+4\cos^2 t\vec{k}, 0\le t\le 2\pi$. We have,

$$\begin{array}{l}\frac{dr}{dt}=-2\sin t\vec{i}+2\cos t\vec{j}-8\cos t\sin t\vec{k}\\ F(t)=2\sin t\vec{i}+4\vec{j}\end{array}\quad\rightarrow\quad F\circ\frac{dr}{dt}=-4\sin^2 t+8\cos t=2\cos 2t-2+8\cos t$$

The work done by F over the curve C is

$$W=\int_0^{2\pi}F\circ\frac{dr}{dt}dt=\int_0^{2\pi}(2\cos 2t-2+8\cos t)dt=-4\pi$$