Solution to MATH 116 INTERMEDIATE CALCULUS III
MIDTERM I EXAM
Date: June 20, 2005, Time: 9:00-11:00

Problem 1.
(a) Use the ε-δ definition of limit to verify that

$$\lim_{(x,y)\to(1,-1)} (2x^2 - 4y^2) = -2$$

(b) Find

$$\lim_{(x,y)\to(0,0)} \frac{\sin x \sin^3 y}{1 - \cos(x^2 + y^2)}$$

or explain why it doesn't exist.

Solution.
(a) For a given $\varepsilon > 0$ take $\delta = \min\{1, \frac{\varepsilon}{18}\}$. Then for any (x, y) satisfying $\sqrt{(x-1)^2 + (y+1)^2} < \delta$, we have
1) $0 < x < 2$, $-2 < y < 0$ (since $\delta \le 1$)
2) $|2x^2 - 4y^2 - (-2)| = |(2x^2 - 2) - (4y^2 - 4)| \le 2|x-1|(x+1) + 4|y+1||y-1| \le 6|x-1| + 12|y+1| \le 18\sqrt{(x-1)^2 + (y+1)^2} < 18\delta \le \varepsilon$. It implies that $\lim_{(x,y)\to(1,-1)} (2x^2 - 4y^2) = -2$.

(b) Since

$$\lim_{(x,y)\to(0,0), along y=mx} \frac{\sin x \sin^3 y}{1 - \cos(x^2 + y^2)} = \lim_{x\to 0} \frac{\sin x \sin^3 mx}{1 - \cos(x^2(1 + m^2))} = \lim_{x\to 0} \frac{x^4 m^3}{\frac{x^4(1+m^2)^2}{2}} = \frac{2m^3}{(1 + m^2)^2},$$

then by the Two-Path Test the limit does not exist.

Problem 2. Let $f(x, y)$ and its first and second order partial derivatives be continuous functions. Suppose

$$f_{xx} + f_{yy} = 0$$

Consider function $w(t, s) = f(t^2 + s^2, t^2 - s^2)$, where $x = t^2 + s^2$ and $y = t^2 - s^2$. Find

$$w_{tt}(1, 1) + w_{ss}(1, 1) \quad ,$$

if it is given that $f_x(2, 0) = 3$.

Solution By the Chain Rule,
$w_t = f_x \frac{\partial x}{\partial t} + f_y \frac{\partial y}{\partial t} = 2tf_x + 2tf_y$
$w_{tt} = 2f_x + 2t(f_{xx}2t + f_{xy}2t) + 2f_y + 2t(f_{yx}2t + f_{yy}2t) = 2f_x + 2f_y + 4t^2(f_{xx} + f_{yy}) + 4t^2(f_{xy} + f_{yx})$
$w_s = f_x \frac{\partial x}{\partial s} + f_y \frac{\partial y}{\partial s} = 2sf_x - 2sf_y$
$w_{ss} = 2f_x + 2s(f_{xx}2s + f_{xy}(-2s)) - 2f_y - 2s(f_{yx}2s - f_{yy}2s) = 2f_x - 2f_y + 4s^2(f_{xx} + f_{yy}) -$

$4s^2(f_{xy} + f_{yx})$
$w_{tt} + w_{ss} = 4f_x + (4t^2 - 4s^2)(f_{xy} + f_{yx})$
Therefore, $w_{tt} + w_{ss}|_{t=1,s=1} = 4f_x(2,0) = 12$.

Problem 3. Find parametric equations for the line tangent to the curve of intersection of the cylinder $4z = 5\sqrt{16 - x^2}$ and the plane $y = 3$ at the point $Q\left(2, 3, \frac{5\sqrt{3}}{2}\right)$.

Solution. The surfaces $4z = 5\sqrt{16 - x^2}$ and $y = 3$ are level surfaces of $f(x, y, z) = 4z - 5\sqrt{16 - x^2}$ and $g(x, y, z) = y - 3$. The tangent line to the intersection curve at point Q is orthogonal to both $\nabla f|_Q$ and $\nabla g|_Q$. Therefore, the tangent line is parallel to $\nabla f|_q \times \nabla g|_Q$ We have,

$$\nabla f|_Q = \frac{10x}{2\sqrt{16 - x^2}}\vec{i} + 4\vec{k}|_Q = \frac{10}{\sqrt{12}}\vec{i} + 4\vec{k}, \qquad \nabla g|_Q = \vec{j},$$

$$\nabla f|_Q \times \nabla g|_Q = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{10}{\sqrt{12}} & 0 & 4 \\ 0 & 1 & 0 \end{vmatrix} = -4\vec{i} + \frac{10}{\sqrt{12}}\vec{k}$$

Parametric equations of the tangent line are

$$\begin{cases} x = 2 - 4t, \\ y = 3, & -\infty < t < \infty, \\ z = \frac{5\sqrt{3}}{2} + \frac{5}{\sqrt{3}}t \end{cases}$$

Problem 4. Find and classify all the critical points of function $f(x, y) = x^2 - xy + y^3 - y$.

Solution. To find critical points we have to solve simultaneously

$$\begin{cases} f_x = 2x - y = 0 \\ f_y = -x + 3y^2 - 1 = 0 \end{cases} \rightarrow \begin{cases} y = 2x \\ -x + 12x^2 - 1 = 0 \end{cases} \rightarrow \begin{cases} y = 2x \\ x = -\frac{1}{4} \quad or \quad x = \frac{1}{3} \end{cases}$$

Critical points are $(-\frac{1}{4}, -\frac{1}{2})$ and $(\frac{1}{3}, \frac{2}{3})$. To classify critical points we find

$$f_{xx} = 2, \quad f_{yy} = 6y, \quad f_{xy} = -1, \quad f_{xx}f_{yy} - f_{xy}^2 = 12y - 1$$

Since at $(-\frac{1}{4}, -\frac{1}{2})$, $f_{xx}f_{yy} - f_{xy}^2 = -7$, then point $(-\frac{1}{4}, -\frac{1}{2})$ is a saddle point of function $f(x, y)$.
Since at point $(\frac{1}{3}, \frac{2}{3}))$, $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 = 7 > 0$, then $(\frac{1}{3}, \frac{2}{3})$ is a local minimum point of function $f(x, y)$.

Problem 5. Find the shortest distance from the origin to the surface $z^2 = 3 + xy$.

Solution. We have to find the minimum value of $f(x, y, z) = x^2 + y^2 + z^2$ subject to constraint $g(x, y, z) = z^3 - 3 - xy = 0$.

Using the Method of Lagrange Multipliers we have to solve simultaneously the following two equations

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y,z) = 0 \end{cases} \quad \to \quad \begin{cases} 2x = -\lambda y \\ 2y = -\lambda x \\ 2z = 2\lambda z \Longleftrightarrow 2z(1-\lambda) = 0 \\ z^2 = 3 + xy \end{cases}$$

<u>Case 1</u>. $z = 0 \Longrightarrow xy = -3, \quad -\lambda = \frac{2x}{y}, \quad -\lambda = \frac{2y}{x} \Longrightarrow \frac{2x}{y} = \frac{2y}{x} \Longrightarrow x^2 = y^2, \quad xy = -3 \Longrightarrow x = \sqrt{3}, y = -\sqrt{3}, z = 0, \quad or \quad x = -\sqrt{3}, y = \sqrt{3}, z = 0.$

<u>Case 2</u>. $\lambda = 1 \Longrightarrow 2x = -y, \quad 2y = -x, z^2 = 3 + xy \Longrightarrow 2(-2x) = -x \Longrightarrow x = 0, y = 0, z = \pm\sqrt{3}.$

Candidate points are $(-\sqrt{3}, \sqrt{3}, 0)$, $(\sqrt{3}, -\sqrt{3}, 0)$, $(0, 0, \sqrt{3})$ and $(0, 0, -\sqrt{3})$. It is clear that the closest points on the surface to the origin are $(0, 0, \sqrt{3})$ and $(0, 0, -\sqrt{3})$. Therefore, the shortest distance from the origin to the surface is $\sqrt{3}$.